Analysis II: Continuity and Differentiability Sheet 8 HT 2019

1. Taylor's Theorem applies to each of the following functions $f(x)$ at the point $a = 0$ and with $n = 4$. Write down each of the terms in the expansion.

(i) $\sin(x^2) - (\sin x)^2$;

(ii) $e^{\alpha x} \cos \beta x$.

2. Suppose that the real-valued function f is such that the $(n-1)^{\text{th}}$ derivative of f exists and is continuous on $[0, h]$ (where $h > 0$) and the n-th derivative exists on $(0, h)$. Consider the function $G : [0, h] \to \mathbb{R}$ defined by

$$G(t) = F(t) - \left(\frac{h-t}{h}\right)^p F(0),$$

where $F : [0, h] \to \mathbb{R}$ is given by

$$F(t) = f(h) - f(t) - (h-t)f'(t) - \cdots - \frac{(h-t)^{n-1}}{(n-1)!} f^{(n-1)}(t)$$

and p is a constant. By considering the derivative of G and choosing p appropriately, prove that there exist θ_1, θ_2 such that

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + S_n$$

where

$$S_n = \frac{h^n}{n!} f^{(n)}(\theta_1 h) = \frac{h^n}{(n-1)!}(1-\theta_2)^{n-1} f^{(n)}(\theta_2 h)\,.$$

3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is such that both f' and f'' exist for all $x \in \mathbb{R}$. Taylor's Theorem tells us that, for each $a, h \in \mathbb{R}$ there is a $\theta \in (0, 1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a + \theta h)\ .$$

Assume further that on the interval $[0, 2]$ the inequalities $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ hold.

Write down the Taylor expansions of $f(0)$ and $f(2)$ about the point $x \in [0, 2]$, using the above form of Taylor's Theorem, with a remainder involving f''. Hence prove that for all $x \in [0, 2]$ we have $|f'(x)| \leq 2$.

4. Compute the Taylor expansion about 0 for $(1+x)^{-1/2}$, and use it to evaluate

$$\sum_{n=0}^{\infty} \binom{2n}{n} \left(-\frac{6}{25}\right)^n .$$

5. By using Taylor's Theorem and the Identity Theorem, prove that

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n} \frac{(1-\frac{1}{2})(2-\frac{1}{2})\cdots\big((n-1)-\frac{1}{2}\big)}{(n-1)!} x^n$$

for $-1 < x \leq 1$.

6. Suppose that f is twice differentiable on $[a, b]$, and $f'(a) = f'(b) = 0$, show that there is $\xi \in (a, b)$ such that

$$|f''(\xi)| \geq \frac{4}{(b-a)^2} |f(b) - f(a)| .$$

[*Hint: By the triangle inequality to obtain*

$$|f(b) - f(a)| \leq |f(b) - f((b+a)/2)| + |f(a) - f((b+a)/2)|.$$

Apply Taylor's formula to f *at* a *and* b].