Groups and Group Actions, Sheet 3, HT18

Subgroups. Cyclic Groups. hcf and lcm. Equivalence relations.

1. For each part below, provide a group G and a proper, non-trivial subgroup H of G according to the different criteria. Provide a different group G in each case.
(i) G is infinite and H is infinite and cyclic;
(ii) G is infinite and H is finite and cyclic;
(iii) G is infinite and non-Abelian and H is Abelian;
(iv) G is infinite and Abelian and H is infinite and not cyclic;
(v) G is infinite and non-Abelian and H is infinite and cyclic.

2. Let G be a group and H, K subgroups of G. Let $HK = \{hk : h \in H, k \in K\}$.
(i) Show that $H \cap K$ is a subgroup of G.
(ii) Give an example where $H \cup K$ is not a subgroup of G. Justify your answer.
(iii) Give an example where HK is not a subgroup of G. Justify your answer.
(iv) Show that if G is Abelian then HK is a subgroup of G .

3. Which of the following groups are cyclic? Either find a generator or show that no generator exists. For the cyclic groups, determine how many different generators there are.

$$\mathbb{Z}^2; \qquad C_2 \times C_4; \qquad C_3 \times C_4; \qquad \langle (1\,2)(3\,4)(5\,6), (1\,4\,5)(2\,3\,6)\rangle \leqslant S_6; \qquad \langle (1\,2\,3), (4\,5\,6)\rangle \leqslant S_6.$$

4. (i) By considering partitions, calculate the number of equivalence relations on a set with four elements.

(ii) Let $q_0 = 1$ and, for $n \geqslant 1$, let q_n denote the number of equivalence relations of a set X with n elements. By considering the possible equivalence classes of the $(n+1)$th element, show that

$$q_{n+1} = \sum_{k=0}^{n} \binom{n}{k} q_{n-k}.$$

Use this recurrence relation to verify your answer for q_4 from part (i).

5. Let G be a finite group. We define a relation $\sim$ on $G\backslash\{e\} = \{g \in G : g \neq e\}$ by

$$g \sim h \text{ if and only there exists } k \text{ such that } g = h^k.$$

(i) Show that $\sim$ is reflexive and transitive.
(ii) Show that $\sim$ is symmetric if and only if the order of g is prime for every $g \neq e$.

6 Let G be a finite group and H, K subgroups of G. The map $\phi : H \times K \to HK$ is defined by $(h, k) \to hk$.
(i) Show that $|\phi^{-1}(g)| = |H \cap K|$ for any $g \in HK$. Deduce that

$$|HK| = \frac{|H|\,|K|}{|H \cap K|}.$$

(ii) Show that if G is Abelian and $H \cap K = \{e\}$ then HK is isomorphic to $H \times K$.