Groups and Group Actions, Sheet 6, TT18

Quotient Groups. Isomorphism Theorem. Group Actions.

1. By constructing a homomorphism with the appropriate kernel and image and then applying the Isomorphism Theorem, demonstrate the following isomorphisms:

(i) $\mathbb{R}^*/\{\pm 1\} \cong (0, \infty)$.
(ii) $\mathbb{C}^*/\{\pm 1\} \cong \mathbb{C}^*$.
(iii) $\mathbb{C}^*/S^1 \cong (0, \infty)$.
(iv) $\mathbb{R}/\mathbb{Z} \cong S^1$.

2. (i) Explain why there is only one homomorphism from $\mathbb{Z}_5$ to D_8 but five homomorphisms from $\mathbb{Z}_5$ to D_{10}.

(ii) What are the normal subgroups N of S_3? For each N, identify the quotient group S_3/N.

(iii) Hence, or otherwise, explain why there are two homomorphisms from S_3 to $\mathbb{Z}_6$ and six homomorphisms from S_3 to D_8.

3. Show that the group S_n (right) acts on the set of subsets of $\{1, 2, \ldots, n\}$ by $\rho(S, \sigma) = S\sigma$.

Show that there are $n + 1$ orbits, one for each possible value of $|S|$. Show that if $|S| = k$ then $\text{Stab}(S) \cong S_k \times S_{n-k}$.

4. Let S denote the set of possible black-or-white colourings of the edges of an equilateral triangle. Explain why $|S| = 8$.

The triangle's symmetry group D_6 acts naturally on S. How many orbits are there?

By listing an element from each orbit, show that there are 10 orbits if three colours are used.

5. (i) Show that $V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4.

(ii) Let $H = \text{Sym}\{1, 2, 3\} = \{\sigma \in S_4 : 4\sigma = 4\}$. How many left cosets of H are there?

Show that if $h_1, h_2 \in H$ and $h_1 V_4 = h_2 V_4$ then $h_1 = h_2$.

(iii) Deduce that S_4/V_4 is isomorphic to S_3.

(iv) With the aid of diagrams, determine the number of essentially different ways there are of labelling the vertices of a rectangle as $1, 2, 3, 4$. How do these diagrams relate to the cosets of V_4 in S_4?

6. (Optional) (i) Let G be a group and H a normal subgroup of G with $|H| = n$. Further let $g \in G$ be such that gH has order m in G/H.

Show that $\langle g \rangle H = \left\{ g^k h : k \in \mathbb{Z},\, h \in H \right\}$ is a subgroup of G of order mn.

(ii) Use this method to construct subgroups of S_4 of orders 8 and 12.