These work.

1) a) X Banach $Y \subseteq X$ $\underbrace{\text{CLOSED}}_{\text{FORCES } \|\ \|_{X/Y} \text{ TO BE A NORM}}$

X/Y

- X is reflexive $\Leftrightarrow$ X^* is reflexive
- reflexivity is preserved under isomorphism.

$$\begin{tikzcd} X^{**} \arrow[r, "T^{**}\,\cong"] & Y^{**} \\ X \arrow[u, "J_X"] \arrow[r, "T", "\cong"'] & Y \arrow[u, "J_Y"'] \end{tikzcd}$$

$J_Y = T^{**} J_X T^{-1}$ is ONTO WHEN J_X IS ONTO.

Show $\pi^*: (X/Y)^* \to X^*$ is isometric & find Ran π^*

Ran $\pi^* = Y^\circ$: For $f \in (X/Y)^*$ & $y \in Y$ $\quad \pi^*(f)(y) = f(\pi(y)) = 0$

$\therefore$ Ran $\pi^* \subseteq Y^\circ$

Now fix $g \in Y^\circ$. "Define" $f(x+Y) = g(x)$ a well defined linear map $X/Y \to \mathbb{C}$ as $g \in Y^\circ$.

Fix $x \in X$, $y \in Y$ $\quad |f(x+Y)| = |g(x+y)| \leq \|g\| \, \|x+y\|$

Take inf over y $\quad |f(x+Y)| \leq \|g\| \, \|x+Y\|$ so $f \in (X/Y)^*$.

$$\|\pi^* f\| = \sup_{x \in B_X^\circ} |f(\pi(x))| = \|f\|.$$

Exhibit an operator $Y^{**} \xrightarrow[\text{isometrically}]{} Y^{oo}$

let

$$\begin{tikzcd} Y \arrow[r, "S"] \arrow[d, "J_Y"'] & X \arrow[d, "J_X"] \\ Y^{**} \arrow[r, "S^{**}"'] & X^{**} \end{tikzcd}$$

be the inclusion so & $\operatorname{Ran}(S^{**}) = Y^{oo}$.

Show π

$\operatorname{Ran} \pi^* =$

Now

Fix

Take

Σ

$|f(x+y)| \le \|g\|\,\|x+y\|$ so $f \in (X/Y)$.

& $\pi^* f = g$.

Note Y is reflexive $\Leftarrow$ $J_X(Y) = Y^{00}$ ($\&\ J_X(Y) \subseteq Y^{00}$)

Let X be reflexive & $\varphi \in Y^{00} \subseteq X^{**}$ so $\varphi = J_X(x)$ some $x \in X$.

Suppose $x \in X \setminus Y$ so by HB $\exists f \in Y^0$ s.t. $f(x) \neq 0$

$$f(x) = J_X(x)(f) = \varphi(f) = 0 \quad \text{⨳} \quad \therefore x \in Y \ \&\ Y \text{ is reflexive.}$$

As $(X/Y)^* \cong Y^0$ a closed subspace of the reflexive space X^*, it is reflexive. Since X/Y is Banach it is reflexive.

Note Y is reflexive $\Leftarrow$ $J_X(Y) = Y^{\circ\circ}$ $\quad (\& \ J_X(Y) \subseteq Y^{\circ\circ})$

Let X be reflexive & $\varphi \in Y^{\circ\circ} \subseteq X^{**}$ so $\varphi = J_X(x)$ some $x \in X$.

Suppose $x \in X \setminus Y$ so by HB $\exists f \in Y^{\circ}$ s.t. $f(x) \neq 0$

$f(x) = J_X(x)(f) = \varphi(f) = 0$ $\times\!\!\times$ $\therefore x \in Y$ & Y is reflexive.

As $(X/Y)^* \cong Y^{\circ}$ a closed subspace of the reflexive space X^*, it is reflexive. Since X/Y is Banach it is reflexive.

Take $\varphi \in X^{**} \xrightarrow{\pi^{**}} (X/Y)^{**}$

So $\pi^{**}(\varphi) = \varphi \circ \pi^{*} \in (X/Y)^{**}$ so $\exists\, x \in X$ s.t

$\varphi \circ \pi^{*} = J_{X/Y}(x+Y)$ as X/Y reflexive

Goal show $\varphi - J_X(x) \in Y^{00}$ so let $f \in Y^{0}$ & write $f = \pi^{*}(g)$

$g \in (X/Y)^{*}$

$$(\varphi - J_X(x))(f) = \varphi(\pi^{*} g) - f(x)$$

$$= J_{X/Y}(x+Y)(g) - f(x)$$

$$= f(x) - f(x) = 0 \quad \therefore \; \varphi - J_X(x) \in Y^{00}$$

& hence $\varphi - J_X(x) = J_X(y)$ some $y \in Y$ as Y reflexive to $J_X(Y) = Y^{00}$.

$\boxed{f_i - f_j = \text{constant } c_{ij} \text{ on } U_{ij}}$

$T : X \xrightarrow[\text{ucc}]{\text{Banach}} Z$ bounded

- X reflexive & T onto show Z is reflexive
- As $Z \simeq X/\ker T$ which is reflexive by 6 & reflexivity preserved under isos.
- If T is injective, X & Z both reflexive by) Ran T not reflexive

WANT Ran T not closed eg $T : \ell^2 \to \ell^2$

$$T(x_n) = \left(\frac{x_n}{n}\right)$$

B_R ball radius R in X^* is metrisable with

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} |f(x_n) - g(x_n)|$$

where (x_n) is dense in B_X.

$I : (B_R ; w^*) \longrightarrow (B_R, d)$ the identity map. $\therefore I$
cpt by B.A. Hsdf is a homeo. $\Leftarrow$ I is cts.

c) X sep. $Y \subseteq X$ closed & $S : Y \longrightarrow c_0$ is an isomorphism.

i) $\exists R > 0$ & $f_n \in B_R$ s.t. $S(y) = (f_n(y))_{n=1}^{\infty}$

T : X —

· X reflexive

· As Z

· If T is in

WANT

1) a) X BANACH $Y \subseteq X$ $\underbrace{\text{CLOSED.}}$

X/Y

$d(f,g) = \sum_{n=1} \frac{1}{2^n} |f(x_n) - g(x_n)|$

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c) X sep. $Y \subseteq X$ closed & $S : Y \longrightarrow c_0$ is an isomorphism.

i) $\exists R > 0$ & $f_n \in B_R$ s.t. $S(y) = (f_n(y))_{n=1}^{\infty}$

let $p_n \in c_0^*$ be $p_n((x_m)) = x_n$ which has $\|p_n\| = 1$

$p_n \circ S \in Y^*$. let f_n be a HB extn of $p_n \circ S$ to X^* s.t. $\|f_n\| \leq \|S\| =: R$

These work.

1) a) X BANACH $Y \subseteq X$ CLOSED.

X/Y

$M = B_R \cap Y^\circ$.

<u>Claim</u>: $\mathrm{dist}(f_n, M) \longrightarrow 0$

If not $\exists\, \varepsilon > 0$ & $n_1 < n_2 < n_3 < \dots$ $\mathrm{dist}(f_{n_i}, M) \geq \varepsilon$

By cptness of B_R $\exists\, f \in B_R$ s.t. $f_{n_i} \longrightarrow f$ (wrt d)

i.e. $f_{n_i} \longrightarrow f$ weak*

$f(y) = \lim_{i \to \infty} f_{n_i}(y) = 0$ as $S(y) = (f_n(y))_{n=1}^\infty \in c_0$.

$\therefore f \in B_R \cap M^\circ$ ※. $\square$.

$M = B_R \cap Y$.

Claim: $\text{dist}(f_n, M) \longrightarrow 0$

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i.e. $f_{n_i} \longrightarrow f$ weak*

$f(y) = \lim_{i \to \infty} f_{n_i}(y) = 0$ as $S(y) = \left(f_n(y)\right)_{n=1}^{\infty} \in c_0$.

$\therefore f \in B_R \cap M^0$ ※. □.

Find $T : X \longrightarrow c_0$ $Tx = $ "Sx" $x \in Y$ & deduce Y is complemented

by a proj P s.t. $\|P\| \leq 2\|S\|\,\|S^{-1}\|$.

Let $h_n \in B_R \cap Y^0$ s.t. $d(f_n, h_n) \leq \text{dist}(f_n, M) + 1/n \longrightarrow 0$

Note Y is reflexive $\iff J_X(Y) = Y^{00}$ ($\&\ J_X(Y) \subseteq Y^{00}$)

$T(x) = (f_n(x) - h_n(x)) \longrightarrow 0$ as $n \to \infty$ since $d(f_n, h_n) \to 0$
so $f_n - h_n \to 0$ weak*

but $h_n \in Y^\circ$ so $T(y) = (f_n(y)) = S(y)$, for $y \in Y$.

$$P = S^{-1} T : X \to Y \text{ \& has } \|P\| \le \|T\| \, \|S^{-1}\|$$
$$\le 2R \|S^{-1}\| = 2\|S\|\|S^{-1}\|.$$

Yes: Consider $X = c$, $Y = c_0$

$P((x_n)) = (x_n - \lim_{r \to \infty} x_r)$ has $\|P\| = 2$.