

These make.

1) a) X BANACH $Y \subseteq X$ CLOSED.
 X/Y

FORCE $\| \cdot \|_{X/Y}$ TO BE A NORM

- X is reflexive $\Leftrightarrow X^*$ is reflexive
- reflexivity is preserved under isomorphism.

$$\begin{array}{ccc} X^{**} & \xrightarrow{T^{**}} & Y^{**} \\ \uparrow J_X & \cong & \uparrow J_Y \\ X & \xrightarrow{T} & Y \end{array}$$

$$J_Y = T^{**} J_X T^{-1} \quad \begin{array}{l} \text{is ONTO} \\ \text{WHEN } J_X \text{ IS ONTO.} \end{array}$$

Show $\pi^*: (X/Y)^* \rightarrow X^*$ is nonzero & $\ker \pi^*$

$\text{Ran } \pi^* = Y^\circ$: For $f \in (X/Y)^*$ & $y \in Y$ $\pi^*(f)(y) = f(\pi(y)) = 0$

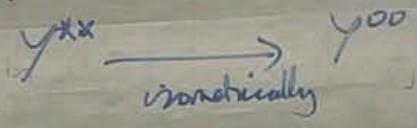
Now fix $g \in Y^\circ$ "Define" $F(x+Y) = g(x)$ a well defined linear map
 $X/Y \rightarrow \mathbb{C}$ as $g \in Y^\circ$.

Fix $x \in X, y \in Y$

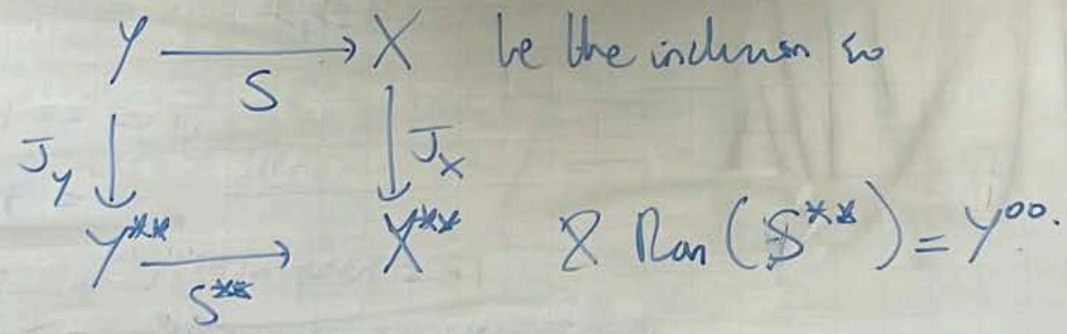
Take any $x \in Y$ $|F(x+Y)| = |g(x+y)| \leq \|g\| \|x+y\|$
 $|F(x+Y)| \leq \|g\| \|x+Y\|$ so $F \in (X/Y)^*$.

$$\|\pi^* f\| = \sup_{x \in B_x^0} |f(\pi(x))| = \|f\|$$

Exhibit an operator



let



Show π^*
Ran π^*

Now \langle

Fix n
 Take i

$\exists \pi^* f = g.$ $\| \pi^* f \| \leq \| g \| \| \pi \|$ so $f \in (X/Y)^*$

Note Y is reflexive $\Leftrightarrow J_X(Y) = Y^{00}$ ($\exists J_X(Y) \subseteq Y^{00}$)

Let X be reflexive & $\varphi \in Y^{00} \subseteq X^{**}$ so $\varphi = J_X(x)$ some $x \in X$.

Suppose $x \in X \setminus Y$ so by HB $\exists f \in Y^0$ s.t. $f(x) \neq 0$

$f(x) = J_X(x)(f) = \varphi(f) = 0$ ~~*~~ $\therefore x \in Y$ & Y is reflexive.

As $(X/Y)^* \cong Y^0$ a closed subspace of the reflexive space X^* , it is reflexive. Since X/Y is Banach it is reflexive.

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Take $\varphi \in X^{**} \xrightarrow{\pi^{**}} (X/Y)^{**}$

So $\pi^{**}(\varphi) = \varphi \circ \pi^* \in (X/Y)^{**}$ so $\exists x \in X$ s.t.

$\varphi \circ \pi^* = J_{X/Y}(x+Y)$ as X/Y reflexive

Goal Show $\varphi - J_X(x) \in Y^{\circ\circ}$ so let $f \in Y^{\circ}$ & write $f = \pi^*(g)$
 $g \in (X/Y)^*$

$$(\varphi - J_X(x))(f) = \varphi(\pi^*g) - f(x)$$

$$= J_{X/Y}(x+Y)(g) - f(x)$$

$$= f(x) - f(x) = 0 \quad \therefore \varphi - J_X(x) \in Y^{\circ\circ}$$

\exists hence $\varphi - J_X(x) = J_X(y)$ some $y \in Y$ as Y reflexive to

$$J_X(y) = Y^{\circ\circ}$$

$$\{f_i - f_j = \text{constant } c_{ij} \text{ on } U_{ij}\}$$

$$T: X \xrightarrow[\text{onto}]{\text{Banach}} Z$$

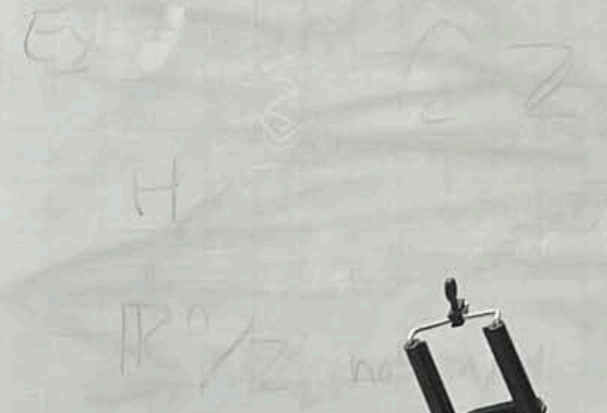
bounded

- X reflexive & T onto show Z is reflexive
- As $Z \cong X / \ker T$ which is reflexive by b & reflexivity preserved under quot.
- If T is injective, X & Z both reflexive by $\Rightarrow \text{Ran } T \text{ not reflexive}$

WANT $\text{Ran } T$ not closed eg

$$T: \ell^2 \rightarrow \ell^2$$

$$T(x_n) = \left(\frac{x_n}{n} \right)$$



B_R ball radius R in X^* is metrisable with

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} |f(x_n) - g(x_n)|$$

where (x_n) is dense in B_X .

$I: (B_R, w^*) \rightarrow (B_R, d)$ the identity map. $\therefore I$
cpt by B.A. $H_{\mathcal{X}}$ is a homeo. $\Leftrightarrow I$ is clo.

c) X sep. $Y \subseteq X$ closed & $S: Y \rightarrow c_0$ is an isomorphism.

i) $\exists R > 0$ & $f_n \in B_R$ s.t. $S(y) = (f_n(y))_{n=1}^{\infty}$

1) a) X BANACH $Y \subseteq X$ CLOSED

$T: X \rightarrow$

X reflexive

$A_3 \subseteq \mathbb{Z}$

If T is lin

WANT

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c) X sep. $Y \subseteq X$ closed & $S: Y \rightarrow c_0$ is an isomorphism.

i) $\exists R > 0$ & $f_n \in B_R$ s.t. $S(y) = (f_n(y))_{n=1}^{\infty}$

let $p_n \in c_0^*$ be $p_n(x_m) = \delta_{nm}$ which has $\|p_n\| = 1$

$p_n \circ S \in Y^*$ let f_n be a H.B. extn of $p_n \circ S$ to X^* s.t. $\|f_n\| \leq \|S\| = R$
 These work.

1) a) X BANACH $Y \subseteq X$ CLOSED

$$M = B_R \cap Y^0$$

Claim: $\text{dist}(f_n, M) \rightarrow 0$

It's not $\exists \epsilon > 0 \ \& \ n_1 < n_2 < n_3 < \dots$

By compactness of $B_R \ \exists f \in B_R$ s.t.

$$f_{n_i} \rightarrow f$$

$$\text{dist}(f_{n_i}, M) \geq \epsilon$$

(wrt d)

$$f(y) = \lim_{i \rightarrow \infty} f_{n_i}(y) = 0$$

$$\text{as } S(y) = (f_{n_i}(y))_{n_i} \in C_0$$

(i.e. $f_{n_i} \rightarrow f$ weak*)

$$\therefore f \in B_R \cap M^0 \quad \text{✗} \quad \square$$

$$M = B_R \cap M$$

Claim: $\text{dist}(f_n, M) \rightarrow 0$

!! not $\exists \varepsilon > 0 \ \& \ n_1 < n_2 < n_3 < \dots$

By openness of $B_R \ \exists f \in B_R$ s.t. $f_{n_i} \rightarrow f$ (wrt d)

$$f(y) = \lim_{i \rightarrow \infty} f_{n_i}(y) = 0 \text{ as } S(y) = (f_{n_i}(y))_{n_i} \in C_0 \text{ (ie. } f_{n_i} \rightarrow f \text{ weak*)}$$

$$\therefore f \in B_R \cap M^\circ \neq \emptyset. \quad \square$$

Find $T: X \rightarrow C_0$ $Tx = \cup Sx$ $x \in Y$ & deduce Y is complete
by using P s.t. $\|P\| \leq 2\|S\| + \|S\|'$

Let $h_n \in B_R \cap Y^\circ$ s.t. $d(f_n, h_n) \leq \text{dist}(f_n, M) + \frac{1}{n} \rightarrow 0$

Note Y is reflexive $\Leftrightarrow J_X(Y) = Y^{\circ\circ}$ ($\& J_X(Y) \subseteq Y^{\circ\circ}$)

these have.

$T(x) = (f_n(x) - h_n(x)) \rightarrow 0$ as $n \rightarrow \infty$ since $d(f_n, h_n) \rightarrow 0$
 so $f_n - h_n \rightarrow 0$ weak*
 but $h_n \in Y^0$ so $T(y) = (f_n(y)) = Sf_y$, for $y \in Y$.

$$P = S^{-1} T : X \rightarrow Y \text{ has } \|P\| \leq \|T\| \|S^{-1}\| \leq 2R \|S^{-1}\| = 2\|S\| \|S^{-1}\|.$$

Yes: Consider $X = C$, $Y = C_0$

$$P(x_n) = (x_n = \lim_{r \rightarrow \infty} x_n) \text{ has } \|P\| = 2.$$

