Groups and Group Actions, Sheet 8, TT18

Orbit-counting Formula. Polyhedral Symmetry Groups.

1. Let G be a finite group with m conjugacy classes. Explain why

$$\sum_{g \in G} |C_G(g)| = m\,|G|\,.$$

Deduce that the probability that two randomly chosen elements of G commute is $m/|G|$. Comment on this formula when G is Abelian.

2. Show that a regular hexagon's edges may be coloured red, white or blue in 92 essentially different ways.

How many ways are possible if an equal number of red, white and blue edges must appear?

3. Let n be an integer such that $n \geqslant 3$ and let S be the set of all ordered triples of strictly positive integers (x, y, z) whose sum is n. Show that S has $\frac{1}{2}(n-1)(n-2)$ elements.

Distinguishing cases, find the number of integer triples (x, y, z) such that

$$x + y + z = n \qquad \text{and} \qquad x \geqslant y \geqslant z > 0.$$

4. (i) Prove that the group of *all* symmetries of a regular tetrahedron is isomorphic to S_4.

(ii) In lectures, it was shown that the group of rotations of a cube is also isomorphic to S_4. Prove that these two groups are not conjugate when considered as subgroups of the group of isometries of three-dimensional space.

5. (i) A frame is made in the shape of a regular tetrahedron and its edges are each coloured with one of n colours. Show that there are

$$\frac{n^6 + 3n^4 + 8n^2}{12}$$

essentially different colourings of this frame.

(ii) Show that the number of different (labelled) colourings of the frame that contain one or more monochromatic triangles is

$$4n^4 - 6n^2 + 3n.$$

[Hint: use the inclusion-exclusion principle.] Explain why these lead to

$$\frac{n^2(n^2+2)}{3}$$

essentially different tetrahedra with one or more monochromatic triangles.