Linear Algebra II Problem Sheet 3, HT 2019

1. Let A be each of the following matrices in turn:

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 4 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

   Find all the eigenvectors of A, determine whether A is diagonalisable (over $\mathbb{R}$) and, if so, find an invertible real matrix P such that $P^{-1}AP$ is diagonal.

2. Define $S : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $S(A) = A^T$. Prove that S has only two distinct eigenvalues and that its eigenvectors span $M_n(\mathbb{R})$.

3. For any polynomial $p(x) = a_0 + a_1x + \cdots + a_kx^k$ and any square matrix A, $p(A)$ is defined as $p(A) = a_0I + a_1A + \cdots + a_kA^k$. Show that if v is any eigenvector of A and $\chi_A(x)$ is the characteristic polynomial of A, then $\chi_A(A)v = 0$, Deduce that if A is diagonalisable then $\chi_A(A)$ is the zero matrix.

4. Let $M = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}$.

   (i) Find a diagonal matrix D and an invertible matrix P such that $M = PDP^{-1}$.

   (ii) Find at least one cube root of M, by observing that if $D = E^3$ then $M = (PEP^{-1})^3$.

   (iii) Express the infinite series $e^M = \sum_{n=0}^{\infty} \frac{1}{n!}M^n$ (where $M^0 = I$) as a 2×2 matrix with entries involving the constant e. (You may assume any general properties of infinite series of matrices that you need.)

5. Let V be a real n dimensional vector space, and $T : V \to V$ be a linear mapping. Show that if λ is the only eigenvalue of T and T is diagonalisable then $T = \lambda I$.

   Now let V be the vector space of real polynomials in x of degree at most d where $d > 0$. Which of the following linear mappings of V into itself are diagonalisable?

   (i) $T_1 : f(x) \mapsto x\frac{df}{dx}$

   (ii) $T_2 : f(x) \mapsto \frac{df}{dx}$

   (iii) $T_3 : f(x) \mapsto f(x+1)$

   (iv) $T_4 : f(x) \mapsto f(-x)$.