Prelims Statistics and Data Analysis – Sheet 3 TT 2019

1. Construct a central $100(1-\alpha)\%$ confidence interval for the unknown parameter μ based on a random sample of size n from a normal distribution with mean μ and variance 1.

If $\alpha = 0.05$ and the length of the interval is to be less than 1, how large must n be? What if the length is to be less than 0.1?

[*In questions 2 and 3, respectively, first assume that it is permissible to replace p, θ by $\overline{x}$ to obtain an estimate of the variance of $\widehat{p}$, $\widehat{\theta}$. Can you find confidence intervals without doing this?*]

2. Suppose $X_1, \ldots, X_n$ is a random sample from a Bernoulli distribution with probability $P(X_i = 1) = p$.

(i) What is the variance of X_i? Show that the estimator $\widehat{p} = \overline{X}$ has expectation p and find its variance.

(ii) Using the central limit theorem construct a random variable which has an approximate standard normal distribution and indicate how this can be used to find a $100(1-\alpha)\%$ confidence interval for p.

(iii) Fifty female black ducks from locations in New Jersey were captured and radio-tagged prior to severe winter months. Of these 19 died during the winter. Find a 95% confidence interval for the proportion surviving. Claims made by environmentalists suggested a 50-50 chance of survival. Is this reasonable?

3. Suppose $X_1, \ldots, X_n$ are independent Poisson random variables each with mean θ. Assuming n is large and using the central limit theorem, construct (i) a central confidence interval for θ, and (ii) an upper confidence limit for θ, each with an associated confidence of $1-\alpha$.

4. Let $X_1, \ldots, X_n$ be i.i.d. Uniform$[0, \theta]$. Find the maximum likelihood estimator $\widehat{\theta}$ of θ. Show that the distribution of $\widehat{\theta}/\theta$ does not depend on θ and show that the interval $(\widehat{\theta}, \widehat{\theta}/\alpha^{1/n})$ is a $1-\alpha$ confidence interval for θ.

5. (a) Let X and Y be independent normally distributed random variables with means a and b and variances v and w. State, without proof, the distribution of $\kappa X + \lambda Y$ for $\kappa, \lambda \in \mathbb{R}$.

Consider a random sample $(L_1, R_1), \ldots, (L_n, R_n)$ of eye pressure measurements in the left and right eyes of n patients.

(b) Suppose that L_j and R_j are independent and normally distributed with unknown mean μ and known variance σ^2, for $j = 1, \ldots, n$. Obtain the likelihood function of the sample and derive the maximum likelihood estimator of μ. Construct a $100(1-\alpha)\%$ confidence interval for μ.

(c) Suppose now that for each j, we do not assume that L_j and R_j are independent. Instead we assume that

$$L_j = M_j + D_j \qquad \text{and} \qquad R_j = M_j - D_j$$

for independent M_j and D_j, where $M_j \sim N(\mu, \sigma_1^2)$ and $D_j \sim N(0, \sigma_2^2)$, with $\sigma_1^2 \geqslant \sigma_2^2$ and $\sigma_1^2 + \sigma_2^2 = \sigma^2$. Here μ is unknown but σ_1^2 and σ_2^2 are known.

Find the distribution of L_j and the distribution of R_j, and find the maximum likelihood estimator for μ in terms of L_j and R_j.

Assuming that $\sigma_1^2 = \sigma_2^2 = \sigma^2/2$, find a $100(1-\alpha)\%$ confidence interval for μ.

Assuming now that $\sigma_1^2 > \sigma_2^2$, state qualitatively, without calculations, how this change will affect (i) the maximum likelihood estimator and (ii) the confidence interval.