In this sheet you may assume standard properties of $\sin, \cos, \exp, \log$, *as established in lectures and on Sheet 3.*

1. Discuss the existence of the following improper integrals.

$$\text{(i)} \int_0^1 \frac{dx}{\sqrt{1-x}}, \qquad \text{(ii)} \int_0^1 \frac{dx}{\sin x}, \qquad \text{(iii)} \int_0^\infty \frac{dx}{1+x^{3/2}}, \qquad \text{(iv)} \int_2^\infty \frac{dx}{x \log x}.$$

2. Let $m, n \geqslant 0$ be nonnegative integers. Show that

$$\lim_{\varepsilon \to 0} \varepsilon^{m+1} (\log \varepsilon)^n = 0$$

(*Hint: you may find it helpful to set* $\varepsilon = e^{-t}$ *and to consider the series expansion of* e^t.)

Using induction on n or otherwise, show that the improper integral

$$\int_0^1 x^m (\log x)^n dx$$

exists, and give a closed-form expression for it.

3. Suppose that $f : [1, \infty) \to \mathbb{R}$ is a continuous function with the property that $f(x) \to 0$ as $x \to \infty$.

(i) Show that $\lim_{X \to \infty} \frac{1}{X} \int_1^X f(x) dx = 0$.

(ii) Does the improper integral $\int_1^\infty \frac{f(x)}{x} dx = \lim_{X \to \infty} \int_1^X \frac{f(x)}{x} dx$ necessarily exist?

4. Show that $\frac{1}{x^x}$ is continuous on $[0, 1]$, and that its integral on this range is equal to $\sum_{n=1}^\infty \frac{1}{n^n}$. *Hint: write* $\frac{1}{x^x} = \sum_{n=0}^\infty \frac{1}{n!} (-x \log x)^n$.

5. Does the improper integral $\int_{2\pi}^\infty \frac{\sin x}{x} dx$ exist?

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