Analysis III 2019 **Further Questions**

Here are some extra questions. The general level is a little towards the harder end of the example sheet questions.

Sheet 1 extras

1. Consider the function $g : [0,1] \to \mathbb{R}$ defined as follows. Set $g(x) = \frac{1}{q}$ if $x = \frac{a}{q}$ is a rational in lowest terms, and $g(x) = 0$ if x is irrational. Show that g is integrable.

Hence, or otherwise, show that there are two integrable functions $f, g : [0,1] \to [0,1]$ such that the composition $f \circ g$ is not integrable.

Sheet 2 extras

2. Suppose that f is Riemann integrable on $[-\pi, \pi]$. Prove the "Riemann–Lebesgue lemma", namely that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0.$$

(*Hint: you may assume standard properties of the cosine function. First check the result for step functions*)

3. Show that if $g : [a,b] \to [c,d]$ is an integrable function $f : [c,d] \to \mathbb{R}$ is continuous then the composition $f \circ g$ is integrable. (*Hint: use Sheet 2, Q5.*)

Sheet 3 extras

4. Let f be a strictly increasing continuous function on $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$.

(i) Show that f has a well-defined inverse g such that $f \circ g(x) = g \circ f(x) = x$, and that g is also strictly increasing and continuous.

(ii) If $\mathcal{P}$, $0 = x_0 < x_1 < \cdots < x_n = 1$ is a partition, write

$$S(f;\mathcal{P}) = \sum_{i=1}^{n} \Big(\frac{f(x_{i-1}) + f(x_i)}{2}\Big)(x_i - x_{i-1}).$$

Show that

$$S(f;\mathcal{P}) \to \int_0^1 f$$

as $\mathrm{mesh}(\mathcal{P}) \to 0$.

(iii) If $f(\mathcal{P})$ denotes the partition $0 = f(x_0) < f(x_1) < \cdots < f(x_n) = 1$, show that

$$S(g, f(\mathcal{P})) \to \int_0^1 g$$

as $\mathrm{mesh}(\mathcal{P}) \to 0$.

(iv) Hence, or otherwise, give an expression for

$$\int_0^1 f(x)dx + \int_0^1 g(x)dx.$$

5. Let $u_n(x) = (1-x^2)^2 x^n$. Prove that $\sum_{n=0}^{\infty} u_n$ converges uniformly on $[0,1]$. By integrating term-by-term, evaulate $\sum_{n=1}^{\infty} \frac{1}{n(n+2)(n+4)}$.

Sheet 4 extras

6. Show that $\int_0^1 \frac{\log x}{x-1} dx$ exists. Show, furthermore, that it equals $\pi^2/6$ (you may assume that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$).

7. Show that the improper integral

$$\int_e^{\infty} \frac{dx}{x(\log x)^c}$$

exists if and only if $c > 1$. (*You need not evaluate the integral, and you may assume that* $\sum_{k=1}^{\infty} k^{-c}$ *converges if and only if* $c > 1$.)

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