1. Determine the following limits (you may assume that standard functions are Riemann integrable and that their integrals are as you learned in school: we'll prove this later in the course).

(i)
$$\lim_{n\to\infty} \frac{1}{n}\left(1 + \mathrm{e}^{\frac{1}{\mathrm{n}}} + \mathrm{e}^{\frac{2}{\mathrm{n}}} \dots + \mathrm{e}^{\frac{\mathrm{n}-1}{\mathrm{n}}}\right)$$

(ii)
$$\lim_{n\to\infty} \frac{1}{n^6}\left(1 + 2^5 + \dots + n^5\right)$$

(iii)
$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1+2n}} + \frac{1}{\sqrt{2+2n}} + \frac{1}{\sqrt{3+2n}} + \dots + \frac{1}{\sqrt{3n}}\right)$$

2. Let $a < b$. Suppose that $\mathcal{P}_i$, $i = 1, 2, \dots$ is a sequence of partitions of $[a, b]$ for which $\text{mesh}(\mathcal{P}_i) \not\to 0$. Show that there is a Riemann integrable function on $[a, b]$ and a sequence of Riemann sums such that $\Sigma(f; \mathcal{P}_i, \xi_i) \not\to \int_a^b f$.

3. By using Riemann sums associated to the sequence of partitions $\mathcal{P}_n$ into n equal parts, show from first principles that $\int_0^1 x^2 dx = \frac{1}{3}$.

4. Is every Riemann integrable function a uniform limit of step functions?

5. Show that a bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if the following is true. For every $\varepsilon > 0$, there is a partition $\mathcal{P} : a = x_0 \leqslant x_1 \leqslant \dots \leqslant x_n = b$, such that the total length of all subintervals (x_{i-1}, x_i) on which $\sup_{x\in(x_{i-1},x_i)} f > \inf_{x\in(x_{i-1},x_i)} f + \varepsilon$ is at most ε.

Note: closed intervals replaced by open May 16th. The question is slightly easier with this formulation

ben.green@maths.ox.ac.uk