Part A Quantum Theory, 2017
Problem Sheet 3 (of 4)

1. Define a linear operator R on wave-functions $\psi(x)$ by

$$(R\psi)(x) := \psi(-x),$$

(this is the *parity* operator but we can't call it P as that is momentum.)

Show that R is self-adjoint and that $R^2 = I$, the identity operator.

What are the possible eigenvalues of R and how can the eigenspaces be characterized?

A particle of mass m moves on the x-axis in a potential $V(x)$. Show that if V is an even function (i.e. $V(-x) = V(x)$) then R commutes with the Hamiltonian (i.e. that $RH\psi = HR\psi$ for all $\psi(x)$).

Show that $R\psi$ is an eigenstate of H with energy E iff ψ is. By considering $\psi \pm R\psi$, or otherwise, deduce that there is either an odd or an even eigenstate (or both) with energy E (so without loss of generality eigenstates of H are simultaneously eigenstates of R).

2. Show that for any infinitely differentiable function ψ of $x \in \mathbb{R}$ whose Taylor series converges to ψ one has, for all real t (*not* time!),

$$(\exp(-itP/\hbar)\psi)(x) = \psi(x - t).$$

Deduce that on the subspace of such functions one has

$$(\exp(-itP/\hbar)X(\exp(itP/\hbar) = X - tI,$$

with X the position operator and I the identity operator.

3. (i) Suppose that ψ_1, ψ_2 are eigenvectors of an observable A with distinct eigenvalues α_1, α_2 respectively. Show that α_1, α_2 are real and ψ_1, ψ_2 are orthogonal.

(ii) Show that the expectation value of an observable A in a state ψ is necessarily real. Conversely show that, if $< \psi|A\psi >$ is real for all ψ then A satisfies

$$< \psi_1|A\psi_2 >=< A\psi_1|\psi_2 >$$

for all ψ_1, ψ_2 (so A is self-adjoint).

Play around with $\psi_1 \pm \psi_2$ and $\psi_1 \pm i\psi_2$, or consider

$$\sum_{k=0}^{3} i^{-k} < \psi_1 + i^k\psi_2|A(\psi_1 + i^k\psi_2) > .$$

4. Take the state space to be $\mathcal{H} = \mathbb{C}^3$, so that a wave-function is a 3-component column vector $\psi = (\psi_1(t), \psi_2(t), \psi_3(t))^T$. Find the stationary states of the Hamiltonian defined by

$$H = \hbar\omega \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix}.$$

[*So the time-dependent Schrödinger equation is* $i\hbar\frac{d\psi}{dt} = H\psi$, *and the stationary-state equation is* $H\psi = E\psi$.]

Now solve the time-dependent Schrödinger equation if $\psi(0) = (1,0,0)^T$. What is the probability that the system is again in this state at time t?

5. In a two-dimensional model of the hydrogen atom, the stationary state Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi \ ,$$

where (r,ϕ) are polar coordinates. By separating the equation via $\psi(r,\phi) = R(r)\Phi(\phi)$, show that $\Phi(\phi)$ is a constant linear combination of $e^{il\phi}$ and $e^{-il\phi}$, where l is a non-negative integer. [*Hint: Use the fact that* $\Phi(\phi + 2\pi) = \Phi(\phi)$.]

By further substituting $R(r) = f(r)e^{-\kappa r}$, where $\kappa = \sqrt{-2mE}/\hbar$, show that the radial equation becomes

$$f'' + \left(\frac{1}{r} - 2\kappa\right)f' - \left(\frac{l^2}{r^2} + \frac{\kappa - \beta}{r}\right)f = 0 \ ,$$

where β is a constant you should identify. By substituting a generalized power series expansion for f, of the form $f(r) = \sum_{k=0}^{\infty} a_k r^{k+c}$, argue that $c = l$ for a non-singular wave function, and hence deduce the recurrence relation

$$a_k = \frac{2\kappa(k+l) - \kappa - \beta}{(k+l)^2 - l^2}a_{k-1}$$

in this case.

Hence or otherwise show that the energy levels are of the form $-\nu/(2n+1)^2$, where ν is a positive constant and n is a non-negative integer (*appeal to normalisability to make the series terminate*). What is the degeneracy of each energy level?

Comments and corrections to `hodges@maths.ox.ac.uk`.