Part A Quantum Theory, 2017
Problem Sheet 4 (of 4)

1. The vector $\psi = \psi_n$ is a normalised eigenvector for the energy level $E = E_n = (n + \frac{1}{2})\hbar\omega$ of the harmonic oscillator with Hamiltonian $H = P^2/2m + \frac{1}{2}m\omega^2 X^2$. Show that

$$E = \mathbb{E}_\psi[P^2]/2m + \frac{1}{2}m\omega^2\mathbb{E}_\psi[X^2].$$

By considering $\langle\psi|(P \pm im\omega X)^k\psi\rangle$ for $k = 1, 2$ and using orthogonality properties of eigenvectors, or otherwise, show that

$$\mathbb{E}_\psi[P] = 0 = \mathbb{E}_\psi[X], \qquad \text{and} \qquad \mathbb{E}_\psi[P^2] = m^2\omega^2\mathbb{E}_\psi[X^2] = mE.$$

[*Use section 7.4.2 in the notes.*]

Deduce that

$$\Delta_\psi[P]\Delta_\psi[X] = \frac{E}{\omega},$$

and discuss how this relates to Heisenberg's Uncertainty Principle.

2. Show that angular momentum operators L_i satisfy

$$\Delta_\psi[L_1]\Delta_\psi[L_2] \geq \frac{1}{2}\hbar|\mathbb{E}_\psi[L_3]|,$$

and find conditions which ensure equality.

[*Don't try to solve the resulting differential equations.*]

3. A particle of mass m and charge e moving in 2 dimensions has Hamiltonian

$$H = \frac{1}{2m}((P_1 + \frac{1}{2}eBX_2)^2 + (P_2 - \frac{1}{2}eBX_1)^2),$$

where B is a constant (and we'll suppose $eB \neq 0$).

Show that the energy levels have the form $(n + 1/2)\hbar|eB|/m$

[*Hint: From the form of the answer you should be thinking of the harmonic oscillator; invent new operators P and X proportional to $P_1 + \frac{1}{2}eBX_2$ and $P_2 - \frac{1}{2}eBX_1$, or vice versa, and a suitable ω so that the given Hamiltonian takes the harmonic oscillator form. Your choices should depend on the sign of eB.*]

4. Use the theory of raising and lowering operators $L_1 \pm iL_2$ to obtain an expression for $Y_{\ell,\ell}(\theta,\phi)$. Now find $Y_{\ell m}(\theta,\phi)$ for $(\ell, m) = (1,1), (1,0), (1,-1), (2,2), (2,1)$ and $(2,0)$ (up to multiplicative constants).

[*For this you need, from the lectures, to recall that $L_3 = -i\hbar\partial/\partial\phi$ and*

$$L_+ = L_1 + iL_2 = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right).$$

Since $Y_{\ell,\ell}, Y_{11}$ and Y_{22} are in the kernel of L_+ and are eigenfunctions of L_3 they are easily found; then lower for the rest.]

5. The Laplacian in spherical polar coordinates is

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} + \frac{1}{r^2\sin^2\theta}\left[\left(\sin\theta\frac{\partial}{\partial\theta}\right)^2 + \frac{\partial^2}{\partial\phi^2}\right]\psi.$$

An electron of mass m moves in a potential of the form $-(\hbar^2/ma)r^{-1}$, where a is the Bohr radius.

Show that $(r\sin\theta e^{i\phi})^\ell \exp[-r/((\ell+1)a)]$ satisfies the time-independent Schrödinger equation with energy $-\hbar^2/2m(\ell+1)^2a^2$.

[*This is easier if you write* $\psi = f(r)(\sin\theta e^{i\phi})^\ell$, *get the equation for* f *and then just check that the given* $f(r)$ *works.*]

Show that (with this wave function) the expected value of r^k can be written as

$$\mathrm{E}[r^k] = \frac{\int_0^\infty r^{2(\ell+1)+k}e^{-2r/(\ell+1)a}\,dr}{\int_0^\infty r^{2(\ell+1)}e^{-2r/(\ell+1)a}\,dr}$$

Deduce that

$$\mathbb{E}[r] = (\ell+\frac{3}{2})(\ell+1)a, \text{ and } \quad \mathbb{E}[r^2] = (\ell+2)(\ell+\frac{3}{2})(\ell+1)^2a^2.$$

[*You may assume that*

$$\int_0^\infty r^n e^{-\lambda r}dr = n!/\lambda^{n+1}.]$$

By using Chebyshev's inequality, $P[|S| > \epsilon] \le \mathbb{E}[S^2]/\epsilon^2$, with $S = (\mathbb{E}[r]-r)/\mathbb{E}[r]$, or otherwise, deduce that

$$P\left[\left|1 - \frac{r}{(\ell+\frac{3}{2})(\ell+1)a}\right| > (2\ell+3)^{-\frac{1}{3}}\right] \le (2\ell+3)^{-\frac{1}{3}},$$

and hence that, for large ℓ, the particle is most likely to be found close to a distance $r = (\ell+\frac{3}{2})(\ell+1)a$ from the origin.

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