Metric spaces and complex analysis

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Problem Sheet 1

1. Let (M, d) be a metric space.

   i) Let (x_n) be sequence in M such that $x_n \to x \in M$ and $x_n \to x' \in M$. Show that $x = x'$.
   ii) Using the definition of continuity in terms of open sets, show that if $f\colon R \to S$ and $g\colon S \to T$ are continuous then $g \circ f$ is continuous.
   iii) Show that the function d_2 given by

$$d_2(f_1, f_2) = \left(\int_a^b (f_1 - f_2)^2 \right)^{1/2}$$

   is not a metric on the space of Riemann integrable functions on $[a, b]$.
   iv) If (M, d) is a finite metric space, show that any subset of M is open.

2. Let (M, d) be a metric space. A non-empty subset $S \subseteq M$ is said to be *bounded* if $\{d(s_1, s_2) : s_1, s_2 \in S\}$ is bounded above in $\mathbb{R}$, and we define its *diameter* as

$$\text{diam}(S) = \sup\{d(s_1, s_2) : s_1, s_2 \in S\}.$$

Let A, B be non-empty bounded subsets of M with $A \cap B \neq \emptyset$. Show that $A \cup B$ is bounded and that $\text{diam}(A \cup B) \leq \text{diam}(A) + \text{diam}(B)$. Show also that if $A \subseteq B$ then $\text{diam}(A) \leq \text{diam}(B)$.

3.

   i) Which of the following subsets of $\mathbb{R}$ are open, which open and which neither? (No proofs required.)

$$(-5, 1) \cup (0, \infty); \quad (-\infty, 2]; \quad \{0\}; \quad (0, 2]; \quad \mathbb{R}; \quad \mathbb{Q}; \quad \mathbb{Z}; \quad \emptyset.$$

   ii) Which of the following subsets of $\mathbb{R}^2$ are open, which closed, and which neither? (No proofs required.)

$$[0, 1] \times \{0\}; \quad (0, 1) \times \{0\}; \quad \{(x, y) : 1 < 4x^2 + y^2 < 4\}; \quad \{(x, y) : xy = 1\}; \quad \mathbb{Z} \times \mathbb{R};$$

$$\{(x, y) : x \in \mathbb{Z} \text{ and } y > 0\}; \quad \{(x, y) : \exp(x^2 + y^2) = 1 + (y^3 - x^3)(x^7 + y^7)\}$$

4. Let $C[a, b]$ be the metric space of continuous functions on $[a, b]$ equipped with the supremum metric. Let $C^1[a, b]$ denote the metric space of continuously differentiable functions also equipped with the supremum metric.

   i) Is $C^1[a, b]$ a closed subset of $C[a, b]$?
   ii) Is differentiation $f \mapsto \frac{df}{dx}$ a continuous function from $C^1[a, b]$ to $C[a, b]$?
   iii) Is integration $f \mapsto \int_a^x f$ a continuous function from $C[a, b]$ to itself?

5. Let $S \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. Write S' for the set of limit points of S in $\mathbb{R}^n$.

   i) Determine S' for each of the following subsets of $\mathbb{R}$.

$$(0, 1); \quad \{0\}; \quad \mathbb{R}; \quad \mathbb{Q}; \quad \mathbb{Z}; \quad \emptyset.$$

   ii) Show that $(S')' \subseteq S'$. Can the containment be strict?

6. Let (M, d_M) and (N, d_N) be metric spaces.

   i) Show that $d\colon (M \times N)^2 \to \mathbb{R}$ given by $d((m_1, n_1), (m_2, n_2)) = d_M(m_1, m_2) + d_N(n_1, n_2)$ defines a metric on $M \times N$.
   ii) Show that if $U \subseteq M$ and $V \subseteq N$ are open sets in M and N respectively, then $U \times V$ is open in $(M \times N, d)$. Deduce that if $F \subseteq M$ and $G \subseteq N$ are closed, then $F \times G$ is closed in $M \times N$.
   iii) Suppose that $A \subseteq M$ and $B \subseteq N$. Show that the closure of $A \times B$ in $M \times N$ is $\bar{A} \times \bar{B}$.

7. Let $S = \{(x, y) \in \mathbb{R}^2 : xy > 0\}$. Which of the following sets are *open in* S and which are *closed in* S?

$$S; \quad \{(x, y) \in S : x \geq 1\}; \quad \{(x, y) \in S : x > 0\};$$

$$\{(1 + 1/n, 1) : n \in \mathbb{N}\}; \quad \{(1/n, 1) : n \in \mathbb{N}\}.$$

8. *i*) Let $\alpha \in \mathbb{R}$ be an irrational number. Show that the function $f\colon \mathbb{Q} \to \mathbb{Q}$ is continuous, where f is given by

$$f(x) = \left\{ \begin{array}{cc} x, & x < \alpha; \\ x+1, & x > \alpha. \end{array} \right.$$

ii) Show that there is no invertible continuous function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(-1) = 0, g(0) = 1, g(1) = -1$.

iii) Show that there is a invertible continuous function $h\colon \mathbb{Q} \to \mathbb{Q}$ such that $h(-1) = 0, h(0) = 1, h(1) = -1$.

9. (*Optional*) Let $\mathbb{P}(\mathbb{R}^n)$ be the set of lines (through the origin) in $\mathbb{R}^n$ and let $d\colon \mathbb{P}(\mathbb{R}^n) \times \mathbb{P}(\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ be given by

$$d(L_1, L_2) = \sqrt{1 - \frac{\langle v, w\rangle^2}{\|v\|^2\|w\|^2}}$$

where $v \in L_1$ and $w \in L_2$ are nonzero vectors. Show that d is a metric on $\mathbb{P}(\mathbb{R}^n)$.