Metric spaces and complex analysis

Mathematical Institute, University of Oxford
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Problem Sheet 2

1. Let $(M.d)$ be a metric space and let A and B be subsets of M. Show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$ but that in general $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

2. Let $f\colon M \to N$ be a map between metric spaces. Show that f is continuous if and only if for every $A \subseteq M$ we have $f(\bar{A}) \subseteq \overline{f(A)}$.

3. A topological space is a set X equipped with a collection of subsets $\mathcal{T}$ which is closed under taking finite intersections and arbitrary unions. Show that if $X = \{0, 1\}$ and $\mathcal{T} = \{\emptyset, \{0\}, \{0, 1\}\}$ then $(X, \mathcal{T})$ is a topological space. Is there a metric on X whose open sets are equal to $\mathcal{T}$?

4. Let (X, d_X) and (Y, d_Y) be metric spaces and let $\mathcal{C}(X, Y)$ be the space of continuous bounded functions from X to Y. Define $\delta\colon \mathcal{C}(X, Y)^2 \to \mathbb{R}$ by

$$\delta(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$$

i) Show that δ is a metric.
ii) Show that if Y is complete then $(\mathcal{C}(X, Y), \delta)$ is complete.
iii) Consider now the map $R\colon \mathcal{C}([0, 1], \mathbb{R}) \to \mathcal{C}((0, 1), \mathbb{R})$ which takes a continuous function on $[0, 1]$ to its restriction to $(0, 1)$. Is the image of R closed?

5. Let M be the set of sequences $(x_n)_{n=0}^{\infty}$ where $x_n \in \{0, 1\}$. Define $d\colon M^2 \to \mathbb{R}$ by

$$d((x_n), (y_n)) = \sum_{n \geq 0} \frac{|x_n - y_n|}{2^n}.$$

i) Show that d is a metric on M.
ii) Let U_0 be the set of sequences (x_n) such that $x_0 = 0$. Show that U_0 is open. Deduce that M is disconnected.
iii) Is M complete?
iv) Let $f\colon M \to \mathbb{R}$ be the function given by $f((x_n)) = \sum_{n=0}^{\infty} \frac{x_n}{2^n}$. Is f continuous?

6. Let M be the space of real $n \times n$ matrices and let $\|A\| = \sup_{v:\|v\|=1} \|A(v)\|$, where $v \in \mathbb{R}^n$ runs over the vectors of norm 1.

i) Show that $\|.\|$ is a norm on M.
ii) Suppose that $A \in M$ has $\|A\| < 1$. Show that the map $B \mapsto AB$ is a contraction. Deduce that $I - A$ is invertible.
Hint: Show that for any vector $v \in \mathbb{R}^n$ we have $\|A(v)\| \leq \|A\|.\|v\|$.

7. *i*) Show directly from the definition that a metric space M is connected if and only if every integer-valued continuous function on M is constant.
ii) Show that $H = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ is connected. By considering the function $f(x, y)/x$ show that there are precisely two continuous functions $f\colon H \to \mathbb{R}$ satisfying $f(x, y)^2 = x^2$ for all $(x, y) \in H$.
iii) How many continuous functions $g\colon \mathbb{R}^2 \to \mathbb{R}$ are there satisfying $g(x, y)^2 = x^2$ for all $(x, y) \in \mathbb{R}^2$?

8. *i*) Prove that if U is an open subset of $\mathbb{R}$ and $c \in U$ then $U \backslash \{c\}$ is disconnected.
ii) Show that if $a \in \mathbb{R}^2$ then the set $\mathbb{R}^2 \backslash \{a\}$ is connected.
iii) By considering the restriction of f to $(0, 1)$, or otherwise, show that there is no invertible continuous function $f\colon [0, 1) \to (0, 1)$.
There are bijections between $[0, 1)$ and $(0, 1)$ however – can you construct one?
iv) Show that there are no continuous one-to-one maps from $\mathbb{R}^2$ to $\mathbb{R}$.

9. (*Optional.*) Let A be a connected subset of a metric space X.

i) If C is a closed and open subset of X show that $A \subseteq C$ or $A \cap C = \emptyset$. Hence or otherwise prove that $\bar{A}$ is a connected subset of X.
ii) Define a relation on X by setting $x \sim y$ if and only if there is a connected subset A of X containing $\{x, y\}$. Show that this is an equivalence relation. The equivalence classes are known as the connected components of X. Show that they are closed subsets of X.