Metric spaces and complex analysis

Mathematical Institute, University of Oxford

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Problem Sheet 3

1. Determine which of the following subsets of $\mathbb{R}^2$ are homeomorphic, giving either a homeomorphism in the case where one exists and a proof that one does not exist.

$$[0,1]\times\{0\};\quad \mathbb{R}^2;\quad D(0,1);\quad \bar{D}(0,1);\quad [0,1]\times[0,1];\quad S^1;\quad \mathbb{R}\times\{0\}.$$

2. Show that a subset $C \subseteq \mathbb{R}^n$ is compact if and only if every continuous function $f\colon C\to\mathbb{R}$ is bounded and attains its bounds.

3. Let M be a metric space and $X_1, X_2, \ldots$ an infinite collection of subsets of M. For each of the following statements, give a proof or counterexample.

i) If $X_1, X_2, \ldots, X_k$ are compact then $X_1 \cup X_2 \cup \ldots \cup X_k$ is compact.

ii) If $X_1, X_2, \ldots, X_k$ are connected then $X_1 \cap X_2 \cap \ldots \cap X_k$ is connected.

iii) If $X_1, X_2, \ldots$ are compact then $\bigcup_{k\geq 1} X_k$ is compact.

iv) If $X_1, X_2, \ldots$ are connected and $X_j \cap X_{j+1} \neq \emptyset$ then $\bigcup_{k\geq 1} X_k$ is connected.

4. *i)* Sketch the following subsets of the complex plane:

$$\{z : |z-i| < |z-1|\};\quad \{z : \mathrm{Im}(\frac{z+i}{2i}) < 0\};\quad \{z : \mathrm{Re}(z+1) = |z-1|\};\quad \{e^z : z \in \mathbb{C}\}.$$

ii) Describe geometrically each of the following maps of the complex plane:

$$z \mapsto \bar{z};\quad z \mapsto e^{i\pi/3}.z;\quad z \mapsto \bar{z}+2i;\quad z \mapsto iz+1.$$

The third map is a reflection, say what its invariant line is. The fourth is a rotation, give the angle and centre of rotation.

iii) Which of the following complex sequences converge?

$$(i^n/n);\quad ((-1)^n n/(n+i));\quad (\frac{n^2+in}{n^2+i});\quad (e^{ni}).$$

5. For each region A and function f, sketch the domain A and its image $f(A)$ of A under the given function f.

i)
$$A = \{z = x+iy \in \mathbb{C} : 0 < x < 1, 0 < y < 1\};\quad f(z) = z^2$$

ii)
$$A = \{z = x+iy \in \mathbb{C} : 0 < x < 1, 0 < y < 1\};\quad f(z) = e^z;$$

iii)
$$A = \{z \in \mathbb{C} : -1 < \mathrm{Im}((1+i)z) < 1\};\quad f(z) = 1/z$$

iv)
$$A = \{z \in \mathbb{C} : |z| < 1\};\quad f(z) = (z-1)^{-1}$$

6. *i)* Show from first principles using the algebra of limits that the function $f(z) = |z|^2$ is differentiable only at zero.

ii) Show that a real-valued function on $\mathbb{C}$ is holomorphic if and only if it is constant.

iii) Suppose that $f\colon U \to \mathbb{C}$ is a holomorphic function on an open subset U. Show that if $V = \{\bar{z} : z \in U\}$ then $z \mapsto \overline{f(\bar{z})}$ is a holomorphic function on V. Show also that $z \mapsto f(\bar{z})$ is a holomorphic function on V if and only if f is constant.

7. Let $u\colon U \to \mathbb{R}$ be a harmonic function on an open set $U \subseteq \mathbb{C}$. A *harmonic conjugate* for u is a function $v\colon U \to \mathbb{C}$ such that $u+iv$ is holomorphic.

i) Show that if v is a harmonic conjugate of u then u is a harmonic conjugate of $-v$.

ii) Assume now that U is connected. Show that, if it exists, a harmonic conjugate of u is unique up to a constant.

iii) Find a harmonic conjugate for:

$$x^3 - 3xy^2 \text{ on } \mathbb{C};\quad e^x\sin(y) \text{ on } \mathbb{C};\quad \log(x^2+y^2) \text{ on } \{z : \mathrm{Re}(z) > 0\}.$$

8. (*Optional.*) Let (X,d) be a complete metric space, and suppose that $h\colon X \to X$ is a map compatible with the distance function, that is $d(x,y) = d(f(x), f(y))$. Prove the following:

i) h is injective;

ii) $h(X)$ is a closed subset of X;

iii) if $a \in X$ and $r = \inf\{d(a, h(x)) : x \in X\}$, and the sequence (x_n) of points of X is defined inductively by $x_0 = a$ and $x_{n+1} = h(x_n)$ for $n \geq 0$, then $d(x_n, x_m) \geq r$ for all $n, m \geq 0$ with $n \neq m$.

iv) if (X, d) is compact then h is bijective.

Give an example of a complete metric space (X, d) and an isometry $h\colon X \to X$ which is not a bijection.