12. The Complex Plane: topology and geometry.

For the rest of the course we will study functions on $\mathbb{C}$ the complex plane, focusing on those which satisfy the complex analogue of differentiability. We will thus need the notions of convergence and limits which $\mathbb{C}$ possesses because it is a metric space (in fact normed vector space).

In this regard, the complex plane is just $\mathbb{R}^2$ and we have seen that there are a number of norms on $\mathbb{R}^2$ which give us the same notion of convergence (and open sets). The additional structure of multiplication which we equip $\mathbb{R}^2$ with when we view it as the complex plane however, makes it natural to prefer the Euclidean one $|z| = \sqrt{(Re(z)^2 + Im(z)^2}$. More explicitly, if $z = (a, b)$ and $w = (c, d)$ are vectors in $\mathbb{R}^2$, then we define their product to be

$$z.w = (ac - bd, ad + bc).$$

It is straight-forward, though a bit tedious, to check that this defines an associative, commutative multiplication on $\mathbb{R}^2$ such that every non-zero element has a multiplicative inverse: if $z = (a, b) \neq (0, 0)$ has $z^{-1} = (a, -b)/(a^2+b^2)$. The number $(1, 0)$ is the multiplicative identity (and so is denoted 1) while $(0, 1)$ is denoted i (or j if you're an engineer) and satisfies $i^2 = -1$. Since $(1, 0)$ and $(0, 1)$ form a basis for $\mathbb{R}^2$ we may write any complex number z uniquely in the form $a + ib$ where $a, b \in \mathbb{R}$. We refer to a and b as the *real* and *imaginary* parts of z, and denote them by $\Re(z)$ and $\Im(z)$ or $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively.

Definition 12.1. If $z = (a, b)$ we write $\bar{z} = (a, -b)$ for the *complex conjugate* of z. It is easy to check that $\overline{zw} = \bar{z}.\bar{w}$ and $\overline{z + w} = \bar{z} + \bar{w}$. The Euclidean norm on $\mathbb{R}^2$ is related to the multiplication of complex numbers by the formula $|z| = \sqrt{z\bar{z}}$, which moreover makes it clear that $|zw| = |z||w|$. (We call such a norm *multiplicative*). If $z \neq 0$ then we will also write $\arg(z) \in \mathbb{R}/2\pi\mathbb{Z}$ for the angle z makes with the positive half of the real axis.

Because subsets of the complex plane can have a much richer structure than subsets of the real line, the topological material we developped in the first half of the course will be indespensible in understanding complex differentiable functions. We will need the notions of completeness, compactness, and connectedness, along with the basic notions of open and closed sets.

Definition 12.2. A connected open subset D of the complex plane will be called a *domain*. As we have already seen, an open set in $\mathbb{C}$ is connected if and only if it is path-connected.

We will also use the notations of closure, interior and boundary of a subset of the complex plane. The *diameter* $\operatorname{diam}(X)$ of a set X is $\sup\{|z-w| : z, w \in X\}$. A set is bounded if and only if it has finite diameter. Recall that the Heine-Borel theorem in the case of $\mathbb{R}^2$ ensures that a subset $X \subseteq \mathbb{C}$ is compact (that is, every open covering has a finite subcover) if and only if it is closed and bounded.

When we study the extended complex plane, lines and circles will become interchangeable (in a sense we will later make precise). The following lemma shows that the two loci can be given a uniform description:

Lemma 12.3. *Any line or circle can be described as* $\{z \in \mathbb{C} : |z-a| = k|z-b|\}$*, where* $a, b \in \mathbb{C}$ *and* $k \in (0,1]$ *and* $a \neq b$*. If* $k = 1$ *one obtains a line, while if* $k < 1$ *one obtains a circle. The parameters* a, b, k *are not unique.*

Proof. Let $C_{a,b,k} = \{z \in \mathbb{C} : |z - a| = k|z - b|\}$. First suppose that $k < 1$. Then we have:

$$\begin{aligned}
|z-a| = k|z-b| &\iff |z-a|^2 = k^2|z-b|^2 \\
&\iff z\bar{z} - a\bar{z} - \bar{a}z + a\bar{a} = k^2(z\bar{z} - b\bar{z} - \bar{b}z + b\bar{b}) \\
&\iff (1-k^2)z\bar{z} - (a - k^2 b)\bar{z} - (\bar{a} - k^2\bar{b})z = -a\bar{a} + k^2 b\bar{b} \\
&\iff |z - \frac{(a-k^2b)}{1-k^2}|^2 - \frac{|a|^2 - k^2(a\bar{b} + \bar{a}b) + k^4|b|^2}{(1-k^2)^2} = \frac{k^2|b|^2 - |a|^2}{1-k^2} \\
&\iff |z - \frac{a-k^2b}{1-k^2}|^2 = \frac{k^2(|a|^2 - a\bar{b} - \bar{a}b + |b|^2)}{(1-k^2)^2} \\
&\iff |z - \frac{a-k^2b}{1-k^2}|^2 = \frac{k^2}{(1-k^2)^2}|a-b|^2.
\end{aligned}$$

Thus $C_{a,b,k}$ is a circle of radius $\frac{k}{1-k^2}|a-b|$ and centre $\frac{a-k^2b}{1-k^2}$. If $k = 1$, then $C_{a,b,1}$ is just the locus of points equidistant from a and b, which is clearly a line (explicitly it is the line through $(a+b)/2$ perpendicular to the line through a and b).

We have thus shown that the loci $C_{a,b,k}$ are either lines or circles. Next we show that any line or circle may be described in this form. If L is a line, picking any two points a, b equidistant to L we see that $L = C_{a,b,1}$. Now suppose that C is a circle. If $T \colon \mathbb{C} \to \mathbb{C}$ is the transformation $z \mapsto rz + s$ (where $r \neq 0$), then it is easy to check that $C_{a,b,k} = T(C_{(a-s)/r,(b-s)/r,k})$, thus the set of circles of the from $C_{a,b,k}$ is preserved under the action of the group of affine linear transformations. But since we can transform any circle in $\mathbb{C}$ to any other circle using such transformations, it follows that every circle occurs as a locus $C_{a,b,k}$ for some $a, b \in \mathbb{C}$, $k \in (0, 1)$.

$\square$

Remark 12.4. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in $\mathbb{C}$. The proof of the above Lemma shows that if we take w_0 with $0 < |w_0| < 1$ and let $w_1 = w_0/|w_0|^2$ and $k = |w_0|$, then $S^1 = C_{w_0,w_1,k}$. Thus, just as for lines, the set of parameters (a, b, k) such that $C_{a,b,k}$ corresponds to a particular circle is infinite. The points a and b are said to be in *inversion* with respect to the circle $C = C_{a,b,k}$.

13. Complex differentiability

We begin by recalling one way of defining the derivative of a real-valued function:

Definition 13.1. Suppose that $f : E \to \mathbb{R}$ is a function and for some $r > 0$ we have $(a - r, a + r) \subseteq E$. Then we say that f is differentiable at a if there is a real number α such that for all $z \in U$ we have

$$f(x) = f(a) + \alpha(x - a) + \epsilon(x)|x - a|,$$

where $\epsilon(x) \to \epsilon(a) = 0$ as $x \to a$. If α exists it is unique and we write $\alpha = f'(a)$.

Remark 13.2. Note that rearranging the above equation we have, for $x \neq a$, $|\epsilon(x)| = |\frac{f(x)-f(a)}{x-a} - \alpha|$, thus the condition that $\epsilon(x) \to 0$ as $x \to a$ is equivalent to $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \alpha$. This also shows the uniqueness of α.

The above formulation of the definition of the derivative is a precise formulation of the statement that a function is differentiable at a point a if there is a "best linear approximation", or tangent line, to f near a – that is, the function $x \mapsto f(a) + f'(a).(x - a)$. (The condition that the error term $\epsilon(x)|x - a|$ goes to zero "faster" than x tends to a is the rigorous meaning given to the adjective "best".) This has the advantage that it generalizes immediately to many variables:

Definition 13.3. Suppose that $E \subseteq \mathbb{R}^2$ is an open set, and $f : E \to \mathbb{R}^2$. Then we say that f is differentiable at $a \in E$ if there is a linear map

$T: \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(z) = f(a) + T(z-a) + \epsilon(x)\|z-a\|$$

where $\epsilon(z) \to \epsilon(a) = 0$ as $z \to a$. If α exists it is unique, and we denote it as $Df(a)$ (or sometimes Df_a. It is known as the *total derivative*[27] of f at a.

One can prove the uniqueness of Df_a directly, but it is more illuminating to understand the relation of α to the partial derivatives: If $v \in \mathbb{R}^2$ we define the *directional derivative* of f at a in the direction v to be

$$\lim_{t \to 0} \frac{f(a+t.v) - f(a)}{t},$$

(if this limit exists). When f is differentiable at a with derivative T, then it follows from the definitions that $\frac{f(a+t.v)-f(a)}{t} = T(v) \pm \epsilon(t.v)\|v\| \to T(v)$ as $t \to 0$, so the directional derivative of f at a all exist. In particular if $z = (x, y)$ and we write $f(z) = (u(x,y), v(x,y)))$ the directional derivatives in the direction of the standard basis vectors e_1 and e_2 are just $(\partial_x u, \partial_x v)$ and $(\partial_y u, \partial_y v)$. Thus we see that if T exists then its matrix with respect to the standard basis is just given by

$$\begin{pmatrix} \partial_x u & \partial_y v \\ \partial_x v & \partial_y v \end{pmatrix}$$

that is the matrix of T is just the *Jacobian matrix* of the partial derivatives of f (and hence the total derivative is uniquely determined, as asserted above).

We are now ready to define what it means for $f: U \to \mathbb{C}$ a function on an open subset U of $\mathbb{C}$, to be *complex differentiable*: We simply require that the linear map T is complex linear, or in other words, that T is given by multiplication by a complex number $f'(a)$:

[27] As opposed to the partial derivatives.

Definition 13.4. A function $f : U \to \mathbb{C}$ on an open subset U of $\mathbb{C}$ is differentiable at $a \in U$ if there exists a complex number $f'(a)$ such that

$$f(z) = f(a) + f'(a).(z - a) + \epsilon(z).|z - a|,$$

where as before $\epsilon(z) \to \epsilon(a) = 0$ as $z \to a$.

Since the standard basis corresponds to $\{1, i\}$, the matrix of the linear map given by multiplication by $w = r + is$ is just

$$\begin{pmatrix} r & -s \\ s & r \end{pmatrix}$$

This gives us our first important result about complex differentiability:

Lemma 13.5. *(Cauchy-Riemann equations): If U is an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$, then f is complex differentiable at $a \in U$ if and only if it is real-differentiable and the partial derivatives satisfy the equations:*

$$\partial_x u = \partial_y v, \quad \partial_x v = -\partial_y u.$$

Proof. This follows immediately from the definitions above. Note that it also shows that the complex derivative satisfies $f'(a) = \partial_x f = \partial_x u + i\partial_x v$ and $f'(a) = \frac{1}{i}\partial_y f = \frac{1}{i}(\partial_y u + i\partial_y v)$. $\square$

Remark 13.6. Since the operation of multiplication by a complex number w is a composition of a rotation (by the argument of w) and a dilation (by the modulus of w) the matrix of the corresponding linear map is, up to scalar, a rotation matrix. The Cauchy-Riemann equations just capture this fact for the matrix of the total (real) derivative of a complex differentiable function.

Remark 13.7. Notice that because we can divide by non-zero complex numbers (which of course we cannot do for vectors in $\mathbb{R}^n$ in general, the definition of the complex derivative, just as for the case of a single real variable, can also be written as

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(z)}{z - a},$$

when the limit exists. This allows us to transport all the basic results about real derivatives, such as the product rule and quotient rule, over to the complex setting – the proofs are identical to the real case (except $|.|$ means the modulus of a complex number rather than the absolute value of a real number).

Proposition 13.8. *Let U be an open subset of $\mathbb{C}$ and let f, g be complex-valued functions on U.*

(1) *If f, g are differentiable at $z_0 \in U$ then $f+g$ and fg are differentiable at z_0 with*

$$(f+g)'(z_0) = f'(z_0) + g'(z_0); \quad (f.g)'(z_0) = f'(z_0).g(z_0) + f(z_0).g'(z_0).$$

(2) *If f, g are differentiable at z_0 and $g(z_0) \neq 0$ and $g'(z_0) \neq 0$ then f/g is differentiable at z_0 with*

$$(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g'(z_0)^2}.$$

(3) *If U and V are open subsets of $\mathbb{C}$ and $f \colon V \to U$ and $g \colon U \to \mathbb{C}$ where f is complex differentiable at $z_0 \in V$ and g is complex differentiable at $f(z_0) \in U$ the $g \circ f$ is complex differentiable at z_0 with*

$$(g \circ f)'(z_0) = g'(f(z_0)).f'(z_0).$$

Proof. These are proved in exactly the same way as they are for a function of a single real variable. $\square$

Remark 13.9. Just as for a single real variable, the basic rules of differentiation allow one to check that polynomial functions are differentiable: Using the product rule and induction one sees that z^n has derivative nz^{n-1} for all $n \geq 0$ (as a constant obviously has derivative 0). Then by linearity it follows every polynomial is differentiable.

A subtlety of real-differentiability in many variables is that it is possible for the partial derivatives of a function to exist without the function being differentiable in the sense of Definition 13.3. In most reasonable situations however, the following theorem shows that this does not happen:

Theorem 13.10. *Let U be an open subset of $\mathbb{R}^2$ and $f\colon U \to \mathbb{R}^2$. Let $f(x) = (f_1(x), f_2(x))^t$. If all the partial derivatives of the $\partial_{x_i} f_j$ exist and are continuous at $z_0 \in U$ then f is differentiable at z_0.*

The proof of this (although it is not hard – one only needs the definitions and the single-variable mean-value theorem) is not part of this course. For completeness, a proof is given in the Appendix. Combining this theorem with the Cauchy-Riemann equations gives a criterion for complex-differentiability:

Theorem 13.11. *Suppose that U is an open subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be a function. If f is differentiable as a function of two real variables with continuous partial derivatives satisfying the Cauchy-Riemann equations on U, then f is complex differentiable on U.*

Proof. Since the partial derivatives are continuous, Theorem 13.10 shows that f is differentiable as a function of two real variables, with total derivative given by the matrix of partial derivatives. If f also satisfies the Cauchy-Riemann equations, then by Lemma 13.5 it follows it is complex differentiable as required. $\square$

Example 13.12. The previous theorem allows us to show that the complex logarithm is a holomorphic function – up to the issue that we cannot define it continuously on the whole complex plane! The function $z \mapsto e^z$ is not injective, since $e^{z+2n\pi i} = e^z$ for all $n \in \mathbb{Z}$ thus it cannot have an inverse defined on all of $\mathbb{C}$. However, since $e^{x+iy} = e^x(\cos(y) + i\sin(y))$, it follows that if we pick a ray through the origin, say $B = \{z \in \mathbb{C} : \Im(z) = 0, \Re(z) \leq 0\}$, then we may define $\mathrm{Log}\colon \mathbb{C}\backslash B \to \mathbb{C}$ by setting $\mathrm{Log}(z) = log(|z|) + i\theta$ where $\theta \in (-\pi, \pi]$ is the argument of z. Clearly $e^{\mathrm{Log}(z)} = z$, while $\mathrm{Log}(e^z)$ differs from z by an integer multiple of $2\pi i$.

We claim that Log is complex differentiable: To show this we use Theorem 13.11. Indeed the function $L(x, y) = (\log(\sqrt{x^2 + y^2}), \theta) = (L_1, L_2)$ has

$$\begin{aligned} \partial_x L_1 &= \frac{x}{x^2+y^2}, \quad \partial_y L_1 = \frac{y}{x^2+y^2}, \\ \partial_x L_2 &= -\frac{y}{x^2+y^2}, \quad \partial_y L_2 = \frac{x}{x^2+y^2}. \end{aligned}$$

where in calculating the partial derivatives of L_2 we used that it is equal to $\arctan(y/x)$ in $(-\pi/2, \pi/2)$ (and other similar expressions in the other two quadrants). Examining the formulae we see that the partial derivatives are all continuous, and obey the Cauchy-Riemann equations, so that Log is indeed complex differentiable.

13.1. **Harmonic functions.** Recall that the two-dimensional Laplace operator Δ is the differential operator $\partial_x^2 + \partial_y^2$ (defined on functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ which are twice differentiable in the sense that their partial derivatives are again differentiable). A function which is in the kernel of the Laplace operator is said to be *harmonic*, that is, a function $u\colon D \to \mathbb{R}$ defined on an open subset D of $\mathbb{R}^2$ is harmonic if $\Delta(u) = \partial_x^2 u + \partial_y^2 u = 0$.

If we work over the complex numbers, then the Laplacian can be factorized[28] as

$$\Delta = (\partial_x + i\partial_y)(\partial_x - i\partial_y) = (\partial_x - i\partial_y)(\partial_x + i\partial_y).$$

The two first-order differential operators $\partial_x + i\partial_y$ and $\partial_x - i\partial_y$ are closely related to the Cauchy-Riemann equations, as we now show, which yields an important connection between complex-differentiable functions and harmonic functions.

Definition 13.13. The *Wirtinger* (partial) derivatives are defined to be $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. By the equation above, we have $\Delta = 4\partial_z\partial_{\bar{z}} = 4\partial_{\bar{z}}\partial_z$ (as operators on twice continuously differentiable functions).

Remark 13.14. Notice that, as you study in Differential Equations, to obtain D'Alembert's solution to the one-dimensional wave equation, one factors $\partial_x^2 - \partial_y^2 = (\partial_x - \partial_y)(\partial_x + \partial_y)$, and then performs the change of coordinates $\eta = x + y, \xi = x - y$. Over the complex numbers, the above factorization of Δ shows that we can analyze the Laplacian in a similar way.

Exercise 13.15. Show that if $T\colon \mathbb{C} \to \mathbb{C}$ is any *real linear* map (that is, viewing $\mathbb{C}$ as $\mathbb{R}^2$ we have $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map) then there are unique $a, b \in \mathbb{C}$ such that $T(z) = az + b\bar{z}$. (*Hint: note that the map $z \mapsto az + b\bar{z}$ is $\mathbb{R}$-linear. What matrix does it correspond to as a map from $\mathbb{R}^2$ to itself?*)

Lemma 13.16. *Let U be an open subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$. Then f satisfies the Cauchy-Riemann equations if and only if $\partial_{\bar{z}} f = 0$.*

Proof. Let $f(z) = u(z) + iv(z)$ where u and v are real-valued. Then we have

$$\partial_{\bar{z}} f = (\partial_x + i\partial_y)(u + iv) = (\partial_x u - \partial_y v) + i(\partial_x v + \partial_y u),$$

[28]Acting on functions which are twice continuously differentiable, the two first order factors commute.

thus the result follows by taking real and imaginary parts. □

Corollary 13.17. *Suppose that U is an open subset of $\mathbb{C}$ and $f: U \to \mathbb{C}$ is complex differentiable and $f(z) = u(z) + iv(z)$ are its real and imaginary parts. If u and v are twice continuously*[29] *differentiable then they are harmonic on U. Moreover any function $g: U \to \mathbb{R}$ is harmonic if it is twice continuously differentiable and $\partial_z(g)$ is complex differentiable.*

Proof. The previous Lemma shows that if f is complex differentiable then $\partial_{\bar{z}} f = 0$. Since the Laplacian Δ is equal to $4\partial_z\partial_{\bar{z}}$ it follows that

$$\Delta(\Re(f)) = \Re(\Delta(f)) = \Re(4\partial_z\partial_{\bar{z}}(f)) = 0,$$

so that $\Re(f)$ is harmonic. Similarly we find $\Im(f)$ is harmonic. The final part is also immediate from the fact that $\Delta = 4\partial_{\bar{z}}\partial_z$. □

Remark 13.18. We will shortly see that if $f = u+iv$ is complex differentiable then it is in fact infinitely complex differentiable. Since we have seen that $f' = \partial_x f = \frac{1}{i}\partial_y f$ it follows that u and v are in fact infinitely differentiable so the condition in the previous lemma on the existence and continuity of their second derivatives holds automatically. For a proof of the fact that the mixed partial derivatives of a twice continuously differentiable function are equal, see the Appendix.

Lemma 13.17 motivates the following definition:

Definition 13.19. If $u: \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function, we say that $v: \mathbb{R}^2 \to \mathbb{R}$ is a *harmonic conjugate* of u if $f(z) = u + iv$ is holomorphic.

Notice that if u is harmonic, it is twice differentiable so that its partial derivatives are continuously differentiable. It follows that a function v is a

[29]That is, all of their second partial deriviatives exist and are continuous.

harmonic conjugate precisely if the pair (u, v) satisfy the Cauchy-Riemann equations. Thus provided we can integrate these equations to find v, a harmonic conjugate will exist. We will show later that, at least when the second partial derivatives are continuous, this can always been done locally in the plane.

13.2. **Power series.** Another important family of examples are the functions which arise from power series. We review here the main results about complex power series which were proved in Analysis II last year:

Definition 13.20. Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers. Then we have an associated sequence of polynomials $s_n(z) = \sum_{k=0}^{n} a_k z^k$. Let S be the set on which this sequence converges pointwise, that is

$$S = \{z \in \mathbb{C} : \lim_{n\to\infty} s_n(z) \text{ exists}\}.$$

Note that since $s_n(0) = a_0$ we have $0 \in S$ so in particular S is nonempty. On the set S, we can define a function $s(z) = \lim_n s_n(z) = \sum_{k=0}^{\infty} a_k z^k$ which we call a *power series*. We define the *radius of convergence* R of the power series $\sum_{k\geq 0} a_k z^k$ to be $\sup\{|z| : z \in S\}$ (or ∞ if S is unbounded).

By convention, given any sequence of complex numbers $(c_n)_{n\geq 0}$ we write $\sum_{k=0}^{\infty} c_k z^k$ for the corresponding power series (even though it may be that it converges only for $z = 0$).

We can give an explicit formula for the radius of convergence using the notion of $\limsup$ which we now recall:

Definition 13.21. If $(a_n)_{n\geq 0}$ is a sequence of real numbers, set $s_n = \sup\{a_k : k \geq n\} \in \mathbb{R} \cup \{\infty\}$ (where we take $s_n = \infty$ if $\{a_k : k \geq n\}$ is not bounded above). Then the sequence (s_n) is either constant and equal to ∞ or eventually becomes a decreasing sequence of real numbers. In the first case we set $\limsup_n a_n = \infty$, whereas in the second case we set $\limsup_n a_n = \lim_n s_n$ (which is finite if (s_n) is bounded below, and equal to $-\infty$ otherwise).

Lemma 13.22. *Let $\sum_{k\geq 0} a_k z^k$ be a power series, let S be the subset of $\mathbb{C}$ on which it converges and let R be its radius of convergence. Then we have*

$$B(0,R) \subseteq S \subseteq \bar{B}(0,R).$$

The series converges absolutely on $B(0,R)$ and if $0 \leq r < R$ then it converges uniformly on $\bar{B}(0,r)$. Moreover, we have

$$1/R = \limsup_n |a_n|^{1/n}.$$

Proof. Let $L = \limsup_n |a_n|^{1/n} \in [0,\infty]$. If $L = 0$ then the statement should be understood to say that the radius of convergence R is ∞, while if $L = \infty$ we take $R = 0$. These two cases are in fact similar but easier than the case where $L \in (0,\infty)$, so we will only give the details for the case where L is finite and positive. Let $s_n = \sup\{|a_k|^{1/k} : k \geq n\}$ so that $L = \lim_{n\to\infty} s_n$.

If $0 < s < 1/L$ we can find an $\epsilon > 0$ such that $(L+\epsilon).s = r < 1$. Thus by definition, for sufficiently large n we have $|a_n|^{1/n} \leq s_n < L + \epsilon$ so that if $|z| \leq s$ we have

$$|a_n||z|^n \leq [(L+\epsilon)|z|]^n \leq r^n,$$

and hence by the comparison test, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely and uniformly on $\bar{B}(0,s)$. It follows the power series converges everywhere in $B(0,1/L)$.

On the other hand, if $|z| > 1/L$ we can find an $\epsilon_1 > 0$ such that $|z|(L - \epsilon_1) = r > 1$. But then for all k we have $s_k \geq L$ since (s_n) is decreasing, and hence by the approximation property for each k we can find an $n_k \geq k$ with $|a_{n_k}|^{1/n_k} > s_k - \epsilon_1 \geq L - \epsilon$ and hence $|a_{n_k} z^{n_k}| > r^k$. Thus $|a_n z^n|$ has a subsequence which does not tend to zero, so the series cannot converge. It follows the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is $1/L$ as claimed.

□

The next lemma is a relatively straight-forward consequence of standard algebra of limits style results:

Lemma 13.23. *Let $s(z) = \sum_{k=0}^{\infty} a_k z^k$ and $t(z) = \sum_{k=0}^{\infty} b_k z^k$ be power series with radii of convergence R_1 and R_2 respectively and let $T = min\{R_1, R_2\}$.*

(1) *Let $c_n = \sum_{k+l=n} a_k b_l$, then the power series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence at least T and if $|z| < T$ we have*
$$\sum_{n=0}^{\infty} c_n z^n = s(z)t(z).$$
Thus the product of power series is a power series.

(2) *If $s(z)$ and $t(z)$ are as above, then $\sum_{k=0}^{\infty}(a_k + b_k)z^k$ is a power series which converges to $s(z) + t(z)$ in $B(0, T)$, thus the sum of power series is again a power series.*

Proof. This was established in Prelims Analysis II. Note that T is only a lower bound for the radius of convergence in each case – it is easy to find examples where the actual radius of convergence of the sum or product is strictly larger than T. □

The behaviour of a power series at its radius of convergence is in general a rather complicated phenomenon. The following result, which we shall not prove, gives some information however. Some of the ideas involved in its proof are investigated in Problem Set 4.

Theorem 13.24. *(Abel's theorem:) Suppose that (a_n) is a sequence of complex numbers and $\sum_{n=0}^{\infty} a_n$ exists. Then the series $\sum_{n=0}^{\infty} a_n z^n$ converges for*

$|z| < 1$ *and*

$$\lim_{\substack{r \in (-1,1) \\ r \uparrow 1}} \big(\sum_{n=0}^{\infty} a_n r^n\big) = \sum_{n=0}^{\infty} a_n.$$

Proof. Note that since the series $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 1$ by assumption, its radius of convergence is at least 1, so that the first statement holds. For the second see for example Exercise 15 of Chapter 1 in the book of Stein and Shakarchi. □

Proposition 13.25. *Let $s(z) = \sum_{k \geq 0} a_k z^k$ be a power series, let S be the domain on which it converges, and let R be its radius of convergence. Then power series $t(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$ also has radius of convergence R and on $B(0, R)$ the power series s is complex differentiable with $s'(z) = t(z)$. In particular, it follows that a power series is infinitely complex differentiable within its radius of convergence.*

Proof. This is proved in Prelims Analysis II. An alternative proof is given in Appendix II. □

Example 13.26. The previous Proposition gives us a large supply of complex differentiable functions. For example,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

are all complex differentiable on the whole complex plane (since $R = \infty$ in each case). Note that one can use the above theorem to show that $\cos(z)^2 + \sin(z)^2 = 1$ for all $z \in \mathbb{C}$, but since $\sin(z)$ and $\cos(z)$ are not in general real, this does not imply that $|\sin(z)|$ or $|\cos(z)|$ at most 1. (In fact it is easy to check that they are both unbounded on $\mathbb{C}$). Using what we have already established about power series it is also easy to check that the complex sin function encompases both the real trigonometric and real hyperbolic functions, indeed:

$$\sin(a+ib) = \sin(a)\cosh(b) + i\cos(a)\sinh(b).$$

Example 13.27. Let $s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$. Then $s(z)$ has radius of convergence 1, and in $B(0,1)$ we have $s'(z) = \sum_{n=0}^{\infty} z^n = 1/(1-z)$, thus this power series is a complex differentiable function which extends the function $-\log(1-z)$ on the interval $(-1,1)$ to the open disc $B(0,1) \subset \mathbb{C}$. We will see later that we will not be able to extend the function log to a complex differentiable function on $\mathbb{C}\backslash\{0\}$ – we will only be able to construct a "multi-valued" extension.

Example 13.28. Recall from Prelims Analysis that the binomial theorem generalizes to non-integral exponents $a \in \mathbb{C}$ if we define $\binom{a}{k} = \frac{1}{k!}a.(a-1)\ldots(a-k+1)$. Indeed we then have

$$(1+z)^a = \sum_{k=0}^{\infty} \binom{a}{k} z^k,$$

for all z with $|z| < 1$. Indeed it is easy to see from the ratio test that this series has radius of convergence equal to 1, and then one can check that if $f(z)$ denotes the function given by the series inside $B(0,1)$, then $zf'(z) = af(z)$.

Note that, slightly more generally, we can work with power series centred at an arbitrary point $z_0 \in \mathbb{C}$. Such power series are functions given by an expression of the form

$$f(z) = \sum_{n\geq 0} a_n(z-z_0)^n.$$

All the results we have shown above immediately extend to these more general power series, since if

$$g(z) = \sum_{n\geq 0} a_n z^n,$$

then the function f is obtained from g simply by composing with the translation $z \mapsto z - z_0$. In particular, the chain rule shows that

$$f'(z) = \sum_{n\geq 1} n a_n (z - z_0)^{n-1}.$$

14. Branch cuts

It is often the case that we study a holomorphic function on a domain $D \subseteq \mathbb{C}$ which does not extend to a function on the whole complex plane.

Example 14.1. Consider the square root "function" $f(z) = z^{1/2}$. Unlike the case of real numbers, every complex number has a square root, but just as for the real numbers, there are two possiblities unless $z = 0$. Indeed if $z = x + iy$ and $w = u + iv$ has $w^2 = z$ we see that

$$u^2 - v^2 = x; \quad 2uv = y,$$

and so

$$u^2 = \frac{x + \sqrt{x^2 + y^2}}{2}, v^2 = \frac{y + \sqrt{x^2 + y^2}}{2}.$$

where the requirement that u^2, v^2 are nonnegative determines the signs. Hence taking square roots we obtain the two possible solutions for w satifying $w^2 = z$. (Note it looks like there are four possible sign combinations in the above, however the requirement that $2uv = y$ means the sign of u determines that of v.) In polars it looks simpler: if $z = re^{i\theta}$ then $w = \pm r^{1/2} e^{i\theta/2}$. Indeed this expression gives us a continuous choice of square root except at the positive real axis: for any $z \in \mathbb{C}$ we may write z uniquely as $re^{i\theta}$

where $\theta \in [0, 2\pi)$, and then set $f(z) = r^{1/2}e^{i\theta/2}$. But now for θ small and positive, $f(z) = r^{1/2}e^{i\theta}$ has small positive argument, but if $z = re^{(2\pi-\epsilon)i}$ we find $f(z) = r^{1/2}e^{(\pi-\epsilon/2)i}$, thus $f(z)$ in the first case is just above the positive real axis, while in the second case $f(z)$ is just below the negative real axis. Thus the function f is only continuous on $\mathbb{C}\backslash\{z \in \mathbb{C} : \Im(z) = 0, \Re(z) > 0\}$. Using Theorem 13.11 you can check f is also holomorphic on this domain. The positive real axis is called a *branch cut* for the *multi-valued function* $z^{1/2}$. By chosing different intervals for the argument (such as $(-\pi, \pi]$ say) we can take different cuts in the plane and obtain different *branches* of the function $z^{1/2}$ defined on their complements.

We formalize these concepts as follows:

Definition 14.2. A *multi-valued function* or *multifunction* on a subset $U \subseteq \mathbb{C}$ is a map $f \colon U \to \mathcal{P}(\mathbb{C})$ assigning to each point in U a subset[30] of the complex numbers. A *branch* of f on a subset $V \subseteq U$ is a function $g \colon V \to \mathbb{C}$ such that $g(z) \in f(z)$, for all $z \in V$. We will be interested in branches of multifunctions which are holomorphic.

Remark 14.3. In order to distinguish between multifunctions and functions, it is sometimes useful to introduce some notation: if we wish to consider $z \mapsto z^{1/2}$ as a multifunction, then to emphasize that we mean a multifunction we will write $[z^{1/2}]$. Thus $[z^{1/2}] = \{w \in \mathbb{C} : w^2 = z\}$. Similarly we write $[\mathrm{Log}(z)] = \{w \in \mathbb{C} : e^w = z\}$. This is not a uniform convention in the subject, but is used, for example, in the text of Priestley.

Thus the square root $z \mapsto [z^{1/2}]$ is a multifunction, and we saw above that we can obtain holomorphic branches of it on a cut plane $\mathbb{C}\backslash R$ where

[30]We use the notation $\mathcal{P}(X)$ to denote the *power set* of X, that is, the set of all subsets of X.

$R = \{te^{i\theta} : t \in \mathbb{R}_{\geq 0}\}$. The point here is that both the origin and infinity as "branch points" for the multifunction $[z^{1/2}]$.

Definition 14.4. Suppose that $f \colon U \to \mathcal{P}(\mathbb{C})$ is a multi-valued function defined on an open subset U of $\mathbb{C}$. We say that $z_0 \in U$ is not a branch point of f if there is an open disk[31] $D \subseteq U$ containing z_0 such that there is a holomorphic branch of f defined on D. We say z_0 is a *branch point* otherwise. When $\mathbb{C}\backslash U$ is bounded, we say that f does not have a branch point at ∞ if there is a branch of f defined on $\mathbb{C}\backslash B(0,R) \subseteq U$ for some $R > 0$. Otherwise we say that ∞ is a branch point of f.

A *branch cut* for a multifunction f is a curve in the plane on whose complement we can pick a holomorphic branch of f. Thus a branch cut must contain all the branch points.

Example 14.5. Another important example of a multi-valued function which we have already discussed is the complex logarithm: as a multifunction we have $\mathrm{Log}(z) = \{\log(|z|) + i(\theta + 2n\pi) : n \in \mathbb{Z}\}$ where $z = |z|e^{i\theta}$. To obtain a branch of the multifunction we must make a choice of argument function $\arg \colon \mathbb{C} \to \mathbb{R}$ we may define

$$\mathrm{Log}(z) = \log(|z|) + i\arg(z),$$

which is a continuous function away from the branch cut we chose. By convention, the *principal branch* of Log is defined by taking $\arg(z) \in (-\pi, \pi]$.

Another important class of examples of multifunctions are the *fractional power* multifunctions $z \mapsto [z^\alpha]$ where $\alpha \in \mathbb{C}$: These are given by

$$z \mapsto \exp(\alpha.[\mathrm{Log}(z)]) = \{\exp(\alpha.w) : w \in \mathbb{C}, e^w = z\}$$

[31]In fact any simply-connected domain – see our discussion of the homotopy form of Cauchy's theorem.

Note this is includes the square root multifunction we discussed above, which can be defined without the use of exponential function. Indeed if $\alpha = m/n$ is rational, $m \in \mathbb{Z}, n \in \mathbb{Z}_{>0}$, then $[z^\alpha] = \{w \in \mathbb{C} : w^m = z^n\}$. For $\alpha \in \mathbb{C}\backslash\mathbb{Q}$ however we can only define $[z^\alpha]$ using the exponential function. Clearly from its definition, anytime we choose a branch $L(z)$ of $[Log(z)]$ we obtain a corresponding branch $\exp(\alpha.L(z))$ of $[z^\alpha]$. If $L(z)$ is the principal branch of $[\mathrm{Log}(z)]$ then the corresponding branch of $[z^\alpha]$ is called the *principal branch* of $[z^\alpha]$.

Example 14.6. Let $F(z)$ be the multi-function

$$[(1+z)^\alpha] = \{\exp(\alpha.w) : w \in \mathbb{C}, \exp(w) = 1+z\}.$$

Then within the open ball $B(0,1)$ the power series $s(z) = \sum_{n=0}^\infty \binom{\alpha}{k} z^k$ yields a holomorphic branch of $[(1+z)^\alpha]$. Indeed we have seen that $(1+z)s'(z) = \alpha.s(z)$, and if we take the principal branch $L(z)$ of $[\mathrm{Log}(z)]$ then on $B(0,1)$ we have[32]

$$\frac{d}{dz}(L(s(z))) = s'(z)/s(z) = \alpha/(1+z) = \frac{d}{dz}(\alpha L(1+z))$$

so that $L(s(z)) = \alpha.L(1+z)+c$ for some constant c (as $B(0,1)$ is connected) which by evaluating at $z=0$ we find is zero. Finally, it follows that $s(z) = \exp(\alpha L(1+z))$ so that $s(z) \in [(1+z)^\alpha]$ as required.

Example 14.7. A more interesting example is the function $f(z) = [(z^2-1)^{1/2}]$. Using the principal branch of the square root function, we obtain a branch f_1 of f on the complement of $E = \{z \in \mathbb{C} : z^2 - 1 \in (-\infty, 0]\}$, which one calculates is equal to $(-1,1) \cup i\mathbb{R}$. If we cross either the segment $(-1,1)$ or the imaginary axis, this branch of f is discontinuous.

[32]Any continuous branch $\ell(z)$ of $[\mathrm{Log}(z)]$ is holomorphic where it is defined and satisfies $\exp(\ell(z)) = z$, hence by the chain rule one obtains $\ell'(z) = 1/z$.

To find another branch, note that we may write $f(z) = \sqrt{z-1}\sqrt{z+1}$, thus we can take the principal branch of the square root for each of these factors. More explicity, if we write $z = 1 + re^{i\theta_1} = -1 + se^{i\theta_2}$ where $\theta_1, \theta_2 \in (-\pi, \pi]$ then we get a branch of f given by $f_2(z) = \sqrt{rs}.e^{i(\theta_1+\theta_2)/2}$. Now the factors are discontinuous on $(-\infty, 1]$ and $(\infty, -1]$ respectively, however let us examine the behaviour of their product: If z crosses the negative real axis at $\Im(z) < -1$ then θ_1 and θ_2 both jumps by 2π, so that $(\theta_1 + \theta_2)/2$ jumps by 2π, and hence $\exp((\theta_1 + \theta_2)/2)$ is in fact continuous. On the other hand, if we cross the segment $(-1, 1)$ then only the factor $\sqrt{z-1}$ switches sign, so our branch is discontinuous there. Thus our second branch of f is defined away from the cut $[-1, 1]$.

Example 14.8. The branch points of the complex logarithm are 0 and infinity: indeed if $z_0 \neq 0$ then we can find a half-plane say $H = \{z \in \mathbb{C} : \Im(z) > 0\}$ (where $|a| = 1$) such that $z_0 \in H$. We can chose a continuous choice of argument function on H, and this gives a holomorphic branch of Log defined on H and hence on the disk $B(z_0, r)$ for r sufficiently small. The logarithm also has a branch point at infinity, since we cannot chose a continous argument function on $\mathbb{C}\backslash B(0, R)$ for any $R > 0$. (We will return to this point when discussing the winding number later in the course.)

Note that if $f(z) = [\sqrt{z^2-1}]$ then the second of our branches f_2 discussed above shows that f does not have a branch point at infinity, whereas both 1 and -1 are branch points – as we move in a sufficiently small circle around we cannot make a continuous choice of branch. One can given a rigorous proof of this using the branch f_2: given any branch g of $[\sqrt{z^2-1}]$ defined on $B(1, r)$ for $r < 1$ one proves that $g = \pm f_2$ so that g is not continuous on $B(0, r) \cap (-1, 1)$. See Problem Sheet 4, question 5, for more details.

Example 14.9. A more sophisticated point of view on branch points and cuts uses the theory of Riemann surfaces. As a first look at this theory, consider the multifunction $f(z) = [\sqrt{z^2-1}]$ again. Let $\Sigma = \{(z,w) \in \mathbb{C}^2 : w^2 = z^2 - 1\}$ (this is an example of a Riemann surface). Then we have two maps from Σ to $\mathbb{C}$, projecting along the first and second factor: $p_1(z,w) = z$ and $p_2(z,w) = w$. Now if $g(z)$ is a branch of f, it gives us a map $G\colon \mathbb{C} \to \Sigma$ where $G(z) = (z, g(z))$. If we take $f_2(z) = \sqrt{z-1}\sqrt{z+1}$ (using the principal branch of the square root function in each case, then let $\Sigma_+\{(z, f_2(z)) : z \notin [-1,1]\}$ and $\Sigma_- = \{(z, -f_2(z)) : z \notin [-1,1]\}$, then $\Sigma_+ \cup \Sigma_-$ covers all of Σ apart from the pairs (z,w) where $z \in [-1,1]$. For such z we have $w = \pm i\sqrt{1-z^2}$, and Σ is obtained by gluing together the two copies Σ_+ and Σ_- of the cut plane $\mathbb{C}\backslash[-1,1]$ along the cut locus $[-1,1]$. However, we must examine the discontinuity of g in order to see how gluing works: the upper side of the cut in Σ_+ is glued to the lower side of the cut in Σ_- and similarly the lower side of the cut in Σ_+ is glued to the upper side of Σ_-.

Notice that on Σ we have the (single-valued) function $p_2(z,w) = w$, and any map $q\colon U \to \Sigma$ from an open subset U of $\mathbb{C}$ to Σ such that $p_1 \circ q(z) = z$ gives a branch of $f(z) = \sqrt{z^2-1}$ given by $p_2 \circ q$. Such a function is called a *section* of p_1. Thus the multi-valued function on $\mathbb{C}$ becomes a single-valued function on Σ, and a branch of the multifunction corresponds to a section of the map $p_1\colon \Sigma \to \mathbb{C}$. In general, given a multi-valued function f one can construct a Riemann surface Σ by gluing together copies of the cut complex plane to obtain a surface on which our multifunction becomes a single-valued function.

15. Paths and Integration

Paths will play a crucial role in our development of the theory of complex differentiable functions. In this section we review the notion of a path and define the integral of a continuous function along a path.

15.1. **Paths.**

Recall that a *path* in the complex plane is a continuous function $\gamma\colon [a,b] \to \mathbb{C}$. A path is said to be *closed* if $\gamma(a) = \gamma(b)$. If γ is a path, we will write γ^* for its image, that is

$$\gamma^* = \{z \in \mathbb{C} : z = \gamma(t), \text{ some } t \in [a,b]\}.$$

Although for some purposes it suffices to assume that γ is continuous, in order to make sense of the integral along a path we will require our paths to be (at least piecewise) differentiable. We thus need to define what we mean for a path to be differentiable:

Definition 15.1. We will say that a path $\gamma\colon [a,b] \to \mathbb{C}$ is *differentiable* if its real and imaginary parts are differentiable as real-valued functions. Equivalently, γ is differentiable at $t_0 \in [a,b]$ if

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists, and denote this limit as $\gamma'(t)$. (If $t = a$ or b then we interpret the above as a one-sided limit.) We say that a path is C^1 if it is differentiable and its derivative $\gamma'(t)$ is continuous.

We will say a path is *piecewise* C^1 if it is continuous on $[a,b]$ and the interval $[a,b]$ can be divided into subintervals on each of which γ is C^1. That is, there is a finite sequence $a = a_0 < a_1 < \ldots < a_m = b$ such that $\gamma_{|[a_i,a_{i+1}]}$ is C^1. Thus in particular, the left-hand and right-hand derivatives of γ at a_i $(1 \leq i \leq m-1)$ may not be equal.

Remark 15.2. Note that a C^1 path may not have a well-defined tangent at every point: if $\gamma\colon [a,b] \to \mathbb{C}$ is a path and $\gamma'(t) \neq 0$, then the line $\{\gamma(t) + s\gamma'(t) : s \in \mathbb{R}\}$ is tangent to γ^*, however if $\gamma'(t) = 0$, the image of γ may have no tangent line there. Indeed consider the example of $\gamma\colon [-1,1] \to \mathbb{C}$ given by

$$\gamma(t) = \begin{cases} t^2 & -1 \leq t \leq 0 \\ it^2 & 0 \leq t \leq 1. \end{cases}$$

Since $\gamma'(0) = 0$ the path is C^1, even though it is clear there is no tangent line to the image of γ at 0.

If $s\colon [a,b] \to [c,d]$ is a differentiable map, then we have the following version of the chain rule, which is proved in exactly the same way as the real-valued case. It will be crucial in our definition of the integral of functions $f\colon \mathbb{C} \to \mathbb{C}$ along paths.

Lemma 15.3. *Let $\gamma\colon [c,d] \to \mathbb{C}$ and $s\colon [a,b] \to [c,d]$ and suppose that s is differentiable at t_0 and γ is differentiable at $s_0 = s(t_0)$. Then $\gamma \circ s$ is differentiable at t_0 with derivative*

$$(\gamma \circ s)'(t_0) = s'(t_0).\gamma'(s(t_0)).$$

Proof. Let $\epsilon\colon [c,d] \to \mathbb{C}$ be given by $\epsilon(s_0) = 0$ and

$$\gamma(x) = \gamma(s_0) + \gamma'(s_0)(x - s_0) + (x - s_0)\epsilon(x),$$

(so that this equation holds for all $x \in [c,d]$), then $\epsilon(x) \to 0$ as $x \to s_0$ by the definition of $\gamma'(s_0)$, *i.e.* ϵ is continuous at t_0. Substituting $x = s(t)$ into this we see that for all $t \neq t_0$ we have

$$\frac{\gamma(s(t)) - \gamma(s_0)}{t - t_0} = \frac{s(t) - s(t_0)}{t - t_0}\big(\gamma'(s(t)) + \epsilon(s(t))\big).$$

Now $s(t)$ is continuous at t_0 since it is differentiable there hence $\epsilon(s(t)) \to 0$ as $t \to t_0$, thus taking the limit as $t \to t_0$ we see that

$$(\gamma \circ s)'(t_0) = s'(t_0)(\gamma'(s_0) + 0) = s'(t_0)\gamma'(s(t_0)),$$

as required. □

Definition 15.4. If $\phi\colon [a,b] \to [c,d]$ is continuously differentiable with $\phi(a) = c$ and $\phi(b) = d$, and $\gamma\colon [c,d] \to \mathbb{C}$ is a C^1-path, then setting $\tilde{\gamma} = \gamma \circ \phi$, by Lemma 15.3 we see that $\tilde{\gamma}\colon [a,b] \to \mathbb{C}$ is again a C^1-path with the same image as γ and we say that $\tilde{\gamma}$ is a *reparametrization* of γ.

Definition 15.5. We will say two parametrized paths $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d] \to \mathbb{C}$ are *equivalent* if there is a continuously differentiable bijective function $s\colon [a,b] \to [c,d]$ such that $s'(t) > 0$ for all $t \in [a,b]$ and $\gamma_1 = \gamma_2 \circ s$. It is straight-forward to check that equivalence is indeed an equivalence relation on parametrized paths, and we will call the equivalence classes *oriented curves* in the complex plane. We denote the equivalence class of γ by $[\gamma]$. The condition that $s'(t) > 0$ ensures that the path is traversed in the same direction for each of the parametrizations γ_1 and γ_2. Moreover γ_1 is piecewise C^1 if and only if γ_2 is.

Recall that we saw before (in a general metric space) that any path $\gamma\colon [a,b] \to \mathbb{C}$ has an *opposite* path γ^- and that two paths $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d] \to \mathbb{C}$ with $\gamma_1(b) = \gamma_2(c)$ can be *concatenated* to give a path $\gamma_1 \star \gamma_2$. If $\gamma, \gamma_1, \gamma_2$ are piecewise C^1 then so are γ^- and $\gamma_1 \star \gamma_2$. (Indeed a piecewise C^1 path is precisely a finite concatenation of C^1 paths).

Remark 15.6. Note that if $\gamma\colon [a,b] \to \mathbb{C}$ is piecewise C^1, then by choosing a reparametrization by a function $\psi\colon [a,b] \to [a,b]$ which is strictly increasing and has vanishing derivative at the points where γ fails to be C^1, we can replace γ by $\tilde{\gamma} = \gamma \circ \psi$ to obtain a C^1 path with the same image. For this reason, some texts insist that C^1 paths have everywhere non-vanishing derivative. In this course we will not insist on this. Indeed sometimes it is convenient to consider a *constant* path, that is a path $\gamma\colon [a,b] \to \mathbb{C}$ such that $\gamma(t) = z_0$ for all $t \in [a,b]$ (and hence $\gamma'(t) = 0$ for all $t \in [a,b]$).

Example 15.7. The most basic example of a closed curve is a circle: If $z_0 \in \mathbb{C}$ and $r > 0$ then the path $z(t) = z_0 + re^{2\pi it}$ (for $t \in [0,1]$) is the simple closed path with *positive orientation* encircling z_0 with radius r. The path $\tilde{z}(t) = z_0 + re^{-2\pi it}$ is the simple closed path encircling z_0 with radius r and *negative orientation*.

Another useful path is a line segment: if $a, b \in \mathbb{C}$ then the path $\gamma_{[a,b]}\colon [0,1] \to \mathbb{C}$ given by $t \mapsto a + t(b-a) = (1-t)a + tb$ traverses the line segment from a to b. We denote the corresponding oriented curve by $[a,b]$ (which is consistent with the notation for an interval in the real line). One of the simplest classes of closed paths are triangles: given three points a, b, c, we define the triangle, or triangular path, associated to them, to be the concatenation of the associated line segments, that is $T_{a,b,c} = \gamma_{a,b} \star \gamma_{b,c} \star \gamma_{c,a}$.

15.2. **Integration along a path.** To define the integral of a complex-valued function along a path, we first need to be able to integrate functions $F\colon [a,b] \to \mathbb{C}$ on a closed interval $[a,b]$ taking values in $\mathbb{C}$. Last year in Analysis III the Riemann integral was defined for a function on a closed interval $[a,b]$ taking values in $\mathbb{R}$, but it is easy to extend this to functions taking values in $\mathbb{C}$: Indeed we may write $F(t) = G(t) + iH(t)$ where G, H are functions on $[a,b]$ taking real values. Then we say that F is Riemann integrable if both G and H are, and we define:

$$\int_a^b F(t)dt = \int_a^b G(t)dt + i\int_a^b H(t)dt$$

Note that if F is continuous, then its real and imaginary parts are also continuous, and so in particular Riemann integrable[33]. The class of Riemann integrable (real or complex valued) functions on a closed interval is however slightly larger than the class of continuous functions, and this will be useful to us at certain points. In particular, we have the following:

Lemma 15.8. *Let $[a,b]$ be a closed interval and $S \subset [a,b]$ a finite set. If f is a bounded continuous function (taking real or complex values) on $[a,b]\backslash S$ then it is Riemann integrable on $[a,b]$.*

Proof. The case of complex-valued functions follows from the real case by taking real and imaginary parts. For the case of a function $f\colon [a,b]\backslash S \to \mathbb{R}$, let $a = x_0 < x_1 < x_2 < \ldots < x_k = b$ be any partition of $[a,b]$ which includes the elements of S. Then on each open interval (x_i, x_{i+1}) the function f is bounded and continuous, and hence integrable. We may therefore set

$$\int_a^b f(t)dt = \int_a^{x_1} f(t)dt + \int_{x_1}^{x_2} f(t)dt + \ldots \int_{x_{k-1}}^{x_k} f(t)dt + \int_{x_k}^{b} f(t)dt.$$

The standard additivity properties of the integral then show that $\int_a^b f(t)dt$ is independent of any choices. $\square$

Remark 15.9. Note that normally when one speaks of a function f being integrable on an interval $[a,b]$ one assumes that f is defined on all of $[a,b]$. However, if we change the value of a Riemann integrable function f at a finite set of points, then the resulting function is still Riemann integrable and its integral is the same. Thus if one prefers the function f in the previous lemma to be defined on all of $[a,b]$ one can define f to take any values at all on the finite set S.

[33]It is clear this definition extends to give a notion of the integral of a function $f\colon [a,b] \to \mathbb{R}^n$ – we say f is integrable if each of its components is, and then define the integral to be the vector given by the integrals of each component function.

It is easy to check that the Riemann integral of complex-valued functions is complex linear. We also note a version of the triangle inequality for complex-valued functions:

Lemma 15.10. *Suppose that $F\colon [a,b] \to \mathbb{C}$ is a complex-valued function. Then we have*
$$|\int_a^b F(t)dt| \leq \int_a^b |F(t)|dt.$$

Proof. First note that if $F(t) = u(t) + iv(t)$ then $|F(t)| = \sqrt{u^2+v^2}$ so that if F is integrable $|F(t)|$ is also[34]. We may write $\int_a^b F(t)dt = re^{i\theta}$, where $r \in [0,\infty)$ and $\theta \in [0, 2\pi)$. Now taking the components of F in the direction of $e^{i\theta}$ and $e^{i(\theta+\pi/s)} = ie^{i\theta}$, we may write $F(t) = u(t)e^{i\theta} + iv(t)e^{i\theta}$. Then by our choice of θ we have $\int_a^b F(t)dt = e^{i\theta}\int_a^b u(t)dt$, and so
$$|\int_a^b F(t)dt| = |\int_a^b u(t)dt| \leq \int_a^b |u(t)|dt \leq \int_a^b |F(t)|dt,$$
where in the first inequality we used the triangle inequality for the Riemann integral of real-valued functions. □

We are now ready to define the integral of a function $f\colon \mathbb{C} \to \mathbb{C}$ along a piecewise-C^1 curve.

Definition 15.11. If $\gamma\colon [a,b] \to \mathbb{C}$ is a piecewise-C^1 path and $f\colon \mathbb{C} \to \mathbb{C}$, then we define the integral of f along γ to be
$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

In order for this integral to exist in the sense we have defined, we have seen that it suffices for the functions $f(\gamma(t))$ and $\gamma'(t)$ to be bounded and continuous at all but finitely many t. Our definition of a piecewise C^1-path ensures that $\gamma'(t)$ is bounded and continuous away from finitely many points

[34] The simplest way to see this is to use that fact that if ϕ is continuous and f is Riemann integrable, then $\phi \circ f$ is Riemann integrable.

(the boundedness follows from the existence of the left and right hand limits at points of discontinuity of $\gamma'(t)$). For most of our applications, the function f will be continuous on the whole image γ^* of γ, but it will occasionally be useful to weaken this to allow $f(\gamma(t))$ finitely many (bounded) discontinuities.

Lemma 15.12. *If $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise C^1 path and $\tilde{\gamma}\colon [c,d] \to \mathbb{C}$ is an equivalent path, then for any continuous function $f\colon \mathbb{C} \to \mathbb{C}$ we have*

$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz.$$

In particular, the integral only depends on the oriented curve $[\gamma]$.

Proof. Since $\tilde{\gamma}$ is equivalent to γ there is a continuously differentiable function $s\colon [c,d] \to [a,b]$ with $s(c) = a$, $s(d) = b$ and $s'(t) > 0$ for all $t \in [c,d]$. Suppose first that γ is C^1. Then by the chain rule we have

$$\begin{aligned} \int_{\tilde{\gamma}} f(z)dz &= \int_c^d f(\gamma(s(t)))(\gamma \circ s)'(t)dt \\ &= \int_c^d f(\gamma(s(t))\gamma'(s(t))s'(t)dt \\ &= \int_a^b f(\gamma(s))\gamma'(s)ds \\ &= \int_\gamma f(z)dz. \end{aligned}$$

where in the second last equality we used the change of variables formula. If $a = x_0 < x_1 < \ldots < x_n = b$ is a decomposition of $[a,b]$ into subintervals such that γ is C^1 on $[x_i, x_{i+1}]$ for $1 \leq i \leq n-1$ then since s is a continuous increasing bijection, we have a corresponding decomposition of $[c,d]$ given by the points $s^{-1}(x_0) < \ldots < s^{-1}(x_n)$, and we have

$$\begin{aligned}
\int_{\tilde{\gamma}} f(z)dz &= \int_c^d f(\gamma(s(t))\gamma'(s(t))s'(t)dt \\
&= \sum_{i=0}^{n-1} \int_{s^{-1}(x_i)}^{s^{-1}(x_{i+1})} f(\gamma(s(t))\gamma'(s(t))s'(t)dt \\
&= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(\gamma(x))\gamma'(x)dx \\
&= \int_a^b f(\gamma(x))\gamma'(x)dx = \int_\gamma f(z)dz.
\end{aligned}$$

where the third equality follows from the case of C^1 paths established above.

□

Definition 15.13. If $\gamma\colon [a,b] \to \mathbb{C}$ is a C^1 path then we define the *length* of γ to be

$$\ell(\gamma) = \int_a^b |\gamma'(t)|dt.$$

Using the chain rule as we did to show that the integrals of a function $f\colon \mathbb{C} \to \mathbb{C}$ along equivalent paths are equal, one can check that the length of a parametrized path is also constant on equivalence classes of paths, so in fact the above defines a length function for oriented curves. The definition extends in the obvious way to give a notion of length for piecewise C^1-paths. More generally, one can define the integral *with respect to arc-length* of a function $f\colon U \to \mathbb{C}$ such that $\gamma^* \subseteq U$ to be

$$\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|dt.$$

This integral is invariant with respect to C^1 reparametrizations $s\colon [c,d] \to [a,b]$ if we require $s'(t) \neq 0$ for all $t \in [c,d]$ (the condition $s'(t) > 0$ is not necessary because of this integral takes the modulus of $\gamma'(t)$). In particular $\ell(\gamma) = \ell(\gamma^-)$.

The integration of functions along piecewise smooth paths has many of the properties that the integral of real-valued functions along an interval possess. We record some of the most standard of these:

Proposition 15.14. *Let $f, g\colon U \to \mathbb{C}$ be continuous functions on an open subset $U \subseteq \mathbb{C}$ and $\gamma, \eta\colon [a,b] \to \mathbb{C}$ be piecewise-C^1 paths whose images lie in U. Then we have the following:*

1. *(Linearity): For $\alpha, \beta \in \mathbb{C}$,*
$$\int_\gamma (\alpha f(z) + \beta g(z))dz = \alpha \int_\gamma f(z)dz + \beta \int_\gamma g(z)dz.$$
2. *If γ^- denotes the opposite path to γ then*
$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$
3. *(Additivity): If $\gamma \star \eta$ is the concatenation of the paths γ, η in U, we have*
$$\int_{\gamma\star\eta} f(z)dz = \int_\gamma f(z)dz + \int_\eta f(z)dz.$$
4. *(Estimation Lemma.) We have*
$$\Big|\int_\gamma f(z)dz\Big| \leq \sup_{z\in\gamma^*} |f(z)|.\ell(\gamma).$$

Proof. Since f, g are continous, and γ, η are piecewise C^1, all the integrals in the statement are well-defined: the functions $f(\gamma(t))\gamma'(t)$, $f(\eta(t))\eta'(t)$, $g(\gamma(t))\gamma'(t)$ and $g(\eta(t))\eta'(t)$ are all Riemann integrable. It is easy to see that one can reduce these claims to the case where γ is smooth. The first claim is immediate from the linearity of the Riemann integral, while the second claim follows from the definitions and the fact that $(\gamma^-)'(t) = -\gamma'(a+b-t)$. The third follows immediately for the corresponding additivity property of Riemann integrable functions.

For the fourth part, first note that $\gamma([a,b])$ is compact in $\mathbb{C}$ since it is the image of the compact set $[a,b]$ under a continuous map. It follows that the function $|f|$ is bounded on this set so that $\sup_{z\in\gamma([a,b])}|f(z)|$ exists. Thus we have

$$\begin{aligned}\Big|\int_\gamma f(z)dz\Big| &= \Big|\int_a^b f(\gamma(t))\gamma'(t)dt\Big| \\ &\leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt \\ &\leq \sup_{z\in\gamma^*}|f(z)|\int_a^b|\gamma'(t)|dt \\ &= \sup_{z\in\gamma^*}|f(z)|.\ell(\gamma).\end{aligned}$$

where for the first inequality we use the triangle inequality for complex-valued functions as in Lemma 15.10 and the positivity of the Riemann integral for the second inequality. □

Remark 15.15. We give part (4) of the above proposition a name (the "estimation lemma") because it will be very useful later in the course. We will give one important application of it now:

Proposition 15.16. *Let $f_n\colon U\to\mathbb{C}$ be a sequence of continuous functions on an open subset U of the complex plane. Suppose that $\gamma\colon[a,b]\to\mathbb{C}$ is a path whose image is contained in U. If (f_n) converges uniformly to a function f on the image of γ then*

$$\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz.$$

Proof. We have

$$\begin{aligned}\left|\int_\gamma f(z)dz - \int_\gamma f_n(z)dz\right| &= \left|\int_\gamma (f(z)-f_n(z))dz\right| \\ &\leq \sup_{z\in\gamma^*}\{|f(z)-f_n(z)|\}.\ell(\gamma),\end{aligned}$$

by the estimation lemma. Since we are assuming that f_n tends to f uniformly on γ^* we have $\sup\{|f(z) - f_n(z)| : z \in \gamma^*\} \to 0$ as $n \to \infty$ which implies the result. □

Definition 15.17. Let $U \subseteq \mathbb{C}$ be an open set and let $f\colon U \to \mathbb{C}$ be a continuous function. If there exists a differentiable function $F\colon U \to \mathbb{C}$ with $F'(z) = f(z)$ then we say F is a *primitive* for f on U.

The fundamental theorem of calculus has the following important consequence[35]:

Theorem 15.18. *(Fundamental theorem of Calculus): Let $U \subseteq \mathbb{C}$ be a open and let $f\colon U \to \mathbb{C}$ be a continuous function. If $F\colon U \to \mathbb{C}$ is a primitive for f and $\gamma\colon [a, b] \to U$ is a piecewise C^1 path in U then we have*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular the integral of such a function f around any closed path is zero.

Proof. First suppose that γ is C^1. Then we have

$$\begin{aligned}\int_\gamma f(z)dz = \int_\gamma F'(z)dz &= \int_a^b F'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b \frac{d}{dt}(F \circ \gamma)(t)dt \\ &= F(\gamma(b)) - F(\gamma(a)),\end{aligned}$$

where in second line we used a version of the chain rule[36] and in the last line we used the Fundamental theorem of Calculus from Prelims analysis on the real and imaginary parts of $F \circ \gamma$.

[35]You should compare this to the existence of a potential in vector calculus.

[36]See the appendix for a discussion of this – we need a version of the chain rule for a composition of real-differentiable functions $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $g\colon \mathbb{R} \to \mathbb{R}^2$.

If γ is only[37] piecewise C^1, then take a partition $a = a_0 < a_1 < \ldots < a_k = b$ such that γ is C^1 on $[a_i, a_{i+1}]$ for each $i \in \{0, 1, \ldots, k-1\}$. Then we obtain a telescoping sum:

$$\begin{aligned}\int_\gamma f(z) &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \sum_{i=0}^{k-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)dt \\ &= \sum_{i=0}^{k-1} (F(\gamma(a_{i+1})) - F(\gamma(a_i))) \\ &= F(\gamma(b)) - F(\gamma(a)),\end{aligned}$$

Finally, since γ is closed precisely when $\gamma(a) = \gamma(b)$ it follows immediately that the integral of f along a closed path is zero. □

Remark 15.19. If $f(z)$ has finitely many point of discontinuity $S \subset U$ but is bounded near them, and $\gamma(t) \in S$ for only finitely many t, then provided F is continuous and $F' = f$ on $U \backslash S$, the same proof shows that the fundamental theorem still holds – one just needs to take a partition of $[a, b]$ to take account of those singularities along with the singularities of $\gamma'(t)$.

Theorem 15.18 already has an important consequence:

Corollary 15.20. *Let U be a domain and let $f: U \to \mathbb{C}$ be a function with $f'(z) = 0$ for all $z \in U$. Then f is constant.*

Proof. Pick $z_0 \in U$. Since U is path-connected, if $w \in U$, we may find[38] a piecewise C^1-path γ: $[0, 1] \to U$ such that $\gamma(a) = z_0$ and $\gamma(b) = w$. Then

[37]The reason we must be careful about this case is that the Fundamental Theorem of Calculus only holds when the integrand is continuous.

[38]Check that you see that if U is an open subset of $\mathbb{C}$ which is path-connected then any two points can be joined by a piecewise C^1-path.

by Theorem 15.18 we see that

$$f(w) - f(z_0) = \int_\gamma f'(z)dz = 0,$$

so that f is constant as required. □

The following theorem is a kind of converse to the fundamental theorem:

Theorem 15.21. *If U is a domain (i.e. it is open and path connected) and $f: U \to \mathbb{C}$ is a continuous function such that for any closed path in U we have $\int_\gamma f(z)dz = 0$, then f has a primitive.*

Proof. Fix z_0 in U, and for any $z \in U$ set

$$F(z) = \int_\gamma f(z)dz.$$

where $\gamma: [a, b] \to U$ with $\gamma(a) = z_0$ and $\gamma(b) = z$.

We claim that $F(z)$ is independent of the choice of γ. Indeed if γ_1, γ_2 are two such paths, let $\gamma = \gamma_1 \star \gamma_2^-$ be the path obtained by concatenating γ_1 and the opposite γ_2^- of γ_2 (that is, γ traverses the path γ_1 and then goes backward along γ_2). Then γ is a closed path and so, using Proposition 15.14 we have

$$0 = \int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2^-} f(z)dz,$$

hence since $\int_{\gamma_2^-} f(z)dz = -\int_{\gamma_2} f(z)dz$ we see that $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$.

Next we claim that F is differentiable with $F'(z) = f(z)$. To see this, fix $w \in U$ and $\epsilon > 0$ such that $B(w, \epsilon) \subseteq U$ and choose a path $\gamma: [a, b] \to U$ from z_0 to w. If $z_1 \in B(w, \epsilon) \subseteq U$, then the concatenation of γ with the straight-line path $s: [0, 1] \to U$ given by $s(t) = w + t(z - w)$from w to z is a path γ_1 from z_0 to z. It follows that

$$\begin{aligned} F(z_1) - F(w) &= \int_{\gamma_1} f(z)dz - \int_{\gamma} f(z)dz \\ &= (\int_{\gamma} f(z)dz + \int_{s} f(z)dz) - \int_{\gamma} f(z)dz \\ &= \int_{s} f(z)dz. \end{aligned}$$

But then we have for $z_1 \neq w$

$$\begin{aligned} \left|\frac{F(z_1) - F(w)}{z_1 - w} - f(w)\right| &= \left|\frac{1}{z_1 - w}\left(\int_0^1 f(w + t(z_1 - w)(z_1 - w)dt\right) - f(w)\right| \\ &= \left|\int_0^1 (f(w + t(z_1 - w)) - f(w))dt\right| \\ &\leq \sup_{t\in[0,1]} |f(w + t(z_1 - w)) - f(w)| \\ &\to 0 \text{ as } z_1 \to w \end{aligned}$$

as f is continuous at w. Thus F is differentiable at w with derivative $F'(w) = f(w)$ as claimed. □

16. Cauchy's theorem

The key insight into the study of holomorphic functions is Cauchy's theorem, which (somewhat informally) states that if $f\colon U \to \mathbb{C}$ is holomorphic and γ is a path in U whose interior lies entirely in U then $\int_\gamma f(z)dz = 0$. It will follow from this and Theorem 15.21 that, at least locally, every holomorphic function has a primitive. The strategy to prove Cauchy's theorem goes as follows: first show it for the simplest closed contours – triangles. Then use this to deduce the existence of a primitive (at least for certain kinds of sufficiently nice open sets U which are called "star-like") and then use Theorem 15.18 to deduce the result for arbitrary paths in such open subsets. We will discuss more general versions of the theorem later, after we have applied Cauchy's theorem for star-like domains to obtain important theorems on the nature of holomorphic functions. First we recall the definition of a triangular path:

Definition 16.1. A *triangle* or *triangular path* T is a path of the form $\gamma_1 \star \gamma_2 \star \gamma_3$ where $\gamma_1(t) = a + t(b-a)$, $\gamma_2(t) = b + t(c-b)$ and $\gamma_3(t) = c + t(a-c)$ where $t \in [0,1]$ and $a, b, c \in \mathbb{C}$. (Note that if $\{a, b, c\}$ are collinear, then T is a degenerate triangle.) That is, T traverses the boundary of the triangle with vertices $a, b, c \in \mathbb{C}$. The solid triangle $\mathcal{T}$ bounded by T is the region

$$\mathcal{T} = \{t_1 a + t_2 b + t_3 c : t_i \in [0,1], \sum_{i=1}^{3} t_i = 1\},$$

with the points in the interior of $\mathcal{T}$ corresponding to the points with $t_i > 0$ for each $i \in \{1, 2, 3\}$. We will denote by $[a, b]$ the line segment $\{a + t(b-a) : t \in [0,1]\}$, the side of T joining vertex a to vertex b. Whenever it is not evident what the vertices of the triangle T are, we will write $T_{a,b,c}$.

Theorem 16.2. *(Cauchy's theorem for a triangle): Suppose that $U \subseteq \mathbb{C}$ is an open subset and let $T \subseteq U$ be a triangle whose interior is entirely contained in U. Then if $f : U \to \mathbb{C}$ is holomorphic we have*

$$\int_T f(z)dz = 0$$

Proof. The proof proceeds using a version of the "divide and conquer" strategy one uses to prove the Bolzano-Weierstrass theorem. Suppose for the sake of contradiction that $\int_T f(z)dz \neq 0$, and let $I = |\int_T f(z)dz| > 0$. We build a sequence of smaller and smaller triangles T^n around which the integral of f is not too small, as follows: Let $T^0 = T$, and suppose that we have constructed T^i for $0 \leq i < k$. Then take the triangle T^{k-1} and join the midpoints of the edges to form four smaller triangles, which we will denote S_i $(1 \leq i \leq 4)$.

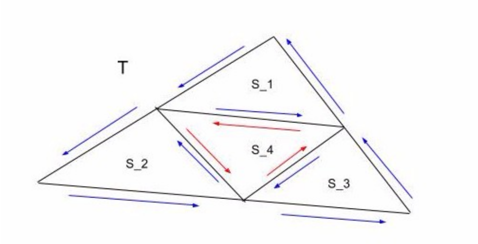

Figure 1. Subdivision of a triangle

Then we have $\int_{T^{k-1}} f(z)dz = \sum_{i=1}^{4} \int_{S_i} f(z)dz$, since the integrals around the interior edges cancel (see Figure 1). In particular, we must have

$$I_k = |\int_{T^{k-1}} f(z)dz| \leq \sum_{i=1}^{4} |\int_{S_i} f(z)dz|,$$

so that for some i we must have $|\int_{S_i} f(z)dz| \geq I_{k-1}/4$. Set T^k to be this triangle S_i. Then by induction we see that $\ell(T^k) = 2^{-k}\ell(T)$ while $I_k \geq 4^{-k}I$.

Now let $\mathcal{T}$ be the solid triangle with boundary T and similarly let $\mathcal{T}^k$ be the solid triangle with boundary T^k. Then we see that $\text{diam}(\mathcal{T}^k) = 2^{-k}\text{diam}(\mathcal{T}) \to 0$, and the sets $\mathcal{T}^k$ are clearly nested. It follows from Lemma 8.6 that there is a unique point z_0 which lies in every $\mathcal{T}^k$. Now by assumption f is holomorphic at z_0, so we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\psi(z),$$

where $\psi(z) \to 0 = \psi(z_0)$ as $z \to z_0$. Note that ψ is continuous and hence integrable on all of U. Now since the linear function $z \mapsto f'(z_0)z + f(z_0)$ clearly has a primitive it follows from Theorem 15.18

$$\int_{T^k} f(z)dz = \int_{T^k} (z - z_0)\psi(z)dz$$

Now since z_0 lies in $\mathcal{T}^k$ and z is on the boundary T^k of $\mathcal{T}^k$, we see that $|z - z_0| \leq \text{diam}(\mathcal{T}^k) = 2^{-k}\text{diam}(T)$. Thus if we set $\eta_k = \sup_{z \in T^k} |\psi(z)|$, it follows by the estimation lemma that

$$I_k = \Big|\int_{T^k}(z-z_0)\psi(z)dz\Big| \leq \eta_k.\mathrm{diam}(T^k)\ell(T^k)$$
$$= 4^{-k}\eta_k.\mathrm{diam}(T).\ell(T).$$

But since $\psi(z) \to 0$ as $z \to z_0$, it follows $\eta_k \to 0$ as $k \to \infty$, and hence $4^k I_k \to 0$ as $k \to \infty$. On the other hand, by construction we have $4^k I_k \geq I > 0$, thus we have a contradiction as required. □

We will later use the following slight extension of this result. If U is an open set and $S \subset U$ is a finite set, then if $f\colon U\backslash S \to \mathbb{C}$ is a continuous function we say that f is *bounded near* $s \in S$ if there is a $\delta > 0$ such that f is bounded on $B(s,\delta)\backslash\{s\}$.

Corollary 16.3. *Suppose that U is open in $\mathbb{C}$ and $S \subset U$ is a finite set. If $f\colon U\backslash S \to \mathbb{C}$ is holomorphic on $U\backslash S$ and is bounded near each $s \in S$. Then if T is any triangle whose interior is entirely contained in U we have*[39] $\int_T f(z)dz = 0$.

Proof. Since f is continuous on $U\backslash S$ and bounded near S, it is bounded on $\mathcal{T}$, so we may take $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathcal{T}$. If the vertices of T are collinear, then the integral $\int_T f(z)dz = 0$ for any $f\colon U \to \mathbb{C}$ which is continuous on $U\backslash S$ and bounded near S as one sees directly from the definition. Otherwise we use induction on $|S|$, the case $|S| = 0$ being established in the previous theorem.

If $|S| > 0$ pick $p \in S$. Let the vertices of T be a, b, c, and first suppose that $p \in \{a,b,c\}$, say $p = a$. Then given $\epsilon > 0$, choose $x \in [a,b]$ and $y \in [a,c]$ such that the triangle $T_{a,x,y}$ with vertices $\{a,x,y\}$ has $\ell(T_{a,x,y}) < \epsilon/M$. Then we

[39]Note that the integral along the triangle is still defined even T contains points in S because f is bounded near the points of S: a continuous (real or complex valued) bounded function g still has a well-defined integral over an interval $[a,b]$ even if it is not defined at a finite subset of $[a,b]$. See Lemma 15.8 for more details.

have

$$\left|\int_T f(z)dz\right| = \left|\int_{T_{a,x,y}} f(z)dz + \int_{T_{x,b,y}} f(z)dz + \int_{T_{y,b,a}} f(z)dz\right|$$
$$= \left|\int_{T_{a,x,y}} f(z)dz\right| \leq \ell(T_{a,x,y}).M < \epsilon.$$

Where the second and third term on the right-hand side of the first line are zero by induction (since they do not contain a by the assumption that a, b, c are not collinear). Since $\epsilon > 0$ was arbitrary, we see that $\int_T f(z)dz = 0$ as required. Now if p is arbitrary, we may apply the above to the triangles $T_{a,b,p}, T_{b,p,c}$ and $T_{c,p,a}$ to conclude that

$$\int_T f(z)dz = \int_{T_{a,b,p}} f(z)dz + \int_{T_{b,p,c}} f(z)dz + \int_{T_{c,p,a}} f(z)dz = 0$$

as required. □

In fact we will see later that this generalization is spurious, in that any function satisfying the hypotheses of the Corollary is in fact holomorphic on all of U, but it will be a key step in our proof of a crucial theorem, the Cauchy integral formula, which will allow us to show that a holomorphic function is in fact infinitely differentiable.

Definition 16.4. Let X be a subset in $\mathbb{C}$. We say that X is *convex* if for each $z, w \in U$ the line segment between z and w is contained in X. We say that X is *star-like* if there is a point $z_0 \in X$ such that for every $w \in X$ the line segment $[z_0, w]$ joining z_0 and w lies in X. We will say that X is star-like with respect to z_0 in this case. Thus a convex subset is thus starlike with respect to every point it contains.

Example 16.5. A disk (open or closed) is convex, as is a solid triangle or rectangle. On the other hand a cross, such as $\{0\} \times [-1, 1] \cup [-1, 1] \times \{0\}$ is star-like with respect to the origin, but is not convex.

Theorem 16.6. *(Cauchy's theorem for a star-like domain): Let U be a star-like domain. The if $f\colon U \to \mathbb{C}$ is holomorphic and $\gamma\colon [a,b] \to U$ is a closed path in U we have*

$$\int_\gamma f(z)dz = 0.$$

Proof. The proof proceeds similarly to the proof of Theorem 15.21: by Theorem 15.18 it suffices to show that f has a primitive in U. To show this, let $z_0 \in U$ be a point for which the line segment from z_0 to every $z \in U$ lies in U. Let $\gamma_z = z_0 + t(z - z_0)$ be a parametrization of this curve, and define

$$F(z) = \int_{\gamma_z} f(\zeta)d\zeta.$$

We claim that F is a primitive for f on U. Indeed pick $\epsilon > 0$ such that $B(z,\epsilon) \subseteq U$. Then if $w \in B(z,\epsilon)$ note that the triangle T with vertices z_0, z, w lies entirely in U by the assumption that U is star-like with respect to z_0. It follows from Theorem 16.2 that $\int_T f(\zeta)d\zeta = 0$, and hence if $\eta(t) = w + t(z - w)$ is the straight-line path going from w to z (so that T is the concatenation of γ_w, η and γ_z^-) we have

$$\begin{aligned}
\Big|\frac{F(z) - F(w)}{z - w} - f(z)\Big| &= \Big|\int_\eta \frac{f(\zeta)}{z - w}d\zeta - f(z)\Big| \\
&= \Big|\int_0^1 f(w + t(z - w))dt - f(z)\Big| \\
&= \Big|\int_0^1 (f(w + t(z - w)) - f(z)dt\Big| \\
&\leq \sup_{t\in[0,1]} |f(w + t(z - w)) - f(z)|,
\end{aligned}$$

which, since f is continuous at w, tends to zero as $w \to z$ so that $F'(z) = f(z)$ as required.

□

Just as we saw for Cauchy's theorem for a triangle, this result can be slightly strengthened as follows:

Corollary 16.7. *If U is a star-like domain and S a finite subset of U. If $f\colon U\backslash S \to \mathbb{C}$ is a holomorphic function which is bounded near each $s \in S$, then $\int_\gamma f(z)dz = 0$ for every closed path $\gamma\colon [a,b] \to U$ for which $\gamma(t) \in S$ for only finitely many $t \in [a,b]$.*

Proof. The condition on γ and the boundedness of f near S ensures that $\int_\gamma f(z)dz$ exists. The proof then proceeds exactly as for the previous theorem, using Corollary 16.3 instead of Theorem 16.2. Note that the proof shows only that $F' = f$ where f is continuous, so potentially not at the points of S. However by Remark 15.19 we just need to check that F is still continuous at $s \in S$. But if $s \in S$ and we may find $\delta, M \in \mathbb{R}_{>0}$ such that $B(s,\delta) \subseteq U$ and $|f(z)| \leq M$ for all $z \in B(s,\delta)\backslash\{s\}$. Then for $z \in B(s,\delta)$, if γ_z denotes the straight-line path from s to z we have

$$|F(z) - F(s)| = |\int_{\gamma_z} f(z)dz| \leq M.\ell(\gamma_z) = M.|z-s|$$

thus F is continuous at s. Since the integral of a function is unaffected if we change the value of the function at finitely many points (and so in particular F' is integrable), we still have

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz = F(\gamma(b)) - F(\gamma(a)),$$

where the second equality holds via a telescoping argument similar to the argument in the proof of Theorem 15.18 for piecewise C^1-paths. Thus the integral of f along any closed path is zero as required. □

Note that our proof of Cauchy's theorem for a star-like domain D proceeded by showing that any holomorphic function on D has a primitive, and hence by the fundamental theorem of calculus its integral around a closed path is zero. This motivates the following definition:

Definition 16.8. We say that a domain $D \subseteq \mathbb{C}$ is *primitive*[40] if any holomorphic function $f \colon D \to \mathbb{C}$ has a primitive in D.

Thus, for example, our proof of Theorem 16.6 shows that all star-like domains are primitive. The following Lemma shows however that we can build many primitive domains which are not star-like.

Lemma 16.9. *Suppose that D_1 and D_2 are primitive domains and $D_1 \cap D_2$ is connected. Then $D_1 \cup D_2$ is primitive.*

Proof. Let $f \colon D_1 \cup D_2 \to \mathbb{C}$ be a holomorphic function. Then $f_{|D_1}$ is a holomorphic function on D_1, and thus it has a primitive $F_1 \colon D_1 \to \mathbb{C}$. Similarly $f_{|D_2}$ has a primitive, F_2 say. But then $F_1 - F_2$ has zero derivative on $D_1 \cap D_2$, and since by assumption $D_1 \cap D_2$ is connected (and thus path-connected) it follows $F_1 - F_2$ is constant, c say, on $D_1 \cap D_2$. But then if $F \colon D_1 \cup D_2 \to \mathbb{C}$ is a defined to be F_1 on D_1 and $F_2 + c$ on D_2 it follows that F is a primitive for f on $D_1 \cup D_2$ as required. $\square$

16.1. **Cauchy's Integral Formula.** We are now almost ready to prove one of the most important consequences of Cauchy's theorem – the integral formula. It is based on the following elementary calculation:

Lemma 16.10. *Let $a \in \mathbb{C}$ and let $\gamma(t) = a + re^{2\pi i t}$ be a parametrization of the circle of radius r centred at a. Then if $w \in B(a, r)$ we have*

$$\int_\gamma \frac{1}{z - w} dz = 2\pi i.$$

[40]This is *not* standard terminology. The reason for this will become clear later.

Proof. Suppose that $|w-a| = \rho < r$. We have

$$\frac{1}{z-w} = \frac{1}{(z-a)-(w-a)} = \frac{1}{z-a}\sum_{n\geq 0}(\frac{w-a}{z-a})^n,$$

where the sum converges uniformly as a function of z for z in the image of γ, since the radius of convergence of $\sum_{k\geq 0} z^k$ is 1. Thus by Lemma 15.16 we see that

$$\begin{aligned}
\int_\gamma \frac{1}{z-w}dz &= \sum_{k\geq 0}(w-a)^k\int_\gamma \frac{1}{(z-a)^{k+1}}dz \\
&= \sum_{k\geq 0}(w-a)^k\int_0^1 r^{-k-1}e^{-2(k+1)\pi it}.(2\pi i r e^{2\pi it})dt \\
&= \sum_{k\geq 0} 2\pi i(w-a)^k\int_0^1 r^{-k}e^{-2k\pi it}dt \\
&= 2\pi i + \sum_{k\geq 1}(w-a)^k r^{-k}(\frac{1-e^{-2k\pi i}}{2k\pi it}) \\
&= 2\pi i
\end{aligned}$$

□

Theorem 16.11. *(Cauchy's Integral Formula.) Suppose that $f\colon U\to\mathbb{C}$ is a holomorphic function on an open set U which contains the disc $\bar{B}(a,r)$. Then for all $w\in B(a,r)$ we have*

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz,$$

where γ is the path $t\mapsto a+re^{2\pi it}$.

Proof. Fix $w\in B(a,r)$ and let $|a-w| = \rho < r$. Consider the function $g(z) = \frac{f(z)-f(w)}{z-w}$ on $U\backslash\{w\}$. Then since f is differentiable at $w\in U$ if we extend g to all of U by defining $g(w) = f'(w)$ it follows that g is continuous on U and, by standard algebraic properties, it is holomorphic on $U\backslash\{w\}$.

Since $\bar{B}(a,r)$ is compact in the open set U, we may find an $R > r$ such that $B(a,R) \subseteq U$. In particular, Corollary 16.7 applies to the function g on the convex set $B(a,R)$, and so $\int_\gamma g(z)dz = 0$. But then we have

$$0 = \int_\gamma \frac{f(z) - f(w)}{z - w} dz = \int_\gamma \frac{f(z)dz}{z - w} - f(w) \int_\gamma \frac{dz}{z - w}.$$

(note that since $w \in B(a,r)$ it does not lie on the image of γ, so that the integrals above all exist). But then by Lemma 16.10 we see that

$$\int_\gamma \frac{f(z)}{z - w} dz = f(w) \int_\gamma \frac{1}{z - w} dz = 2\pi i f(w),$$

and the result follows. □

Remark 16.12. The same result holds for any oriented curve γ for which we can make sense of the notion of the "interior" of the curve γ. We will develop this generalization later using the notion of the *winding number* of a path around a point $w \notin \gamma^*$.

Remark 16.13. Note that the same integral formula also holds if f is only defined on $U \backslash S$ where S is a finite set, provided that f is bounded near the points of S. This follows by applying Corollary 16.7 in place of Theorem 16.6.

Remark 16.14. This formula has many remarkable consequences: note first of all that it implies that if f is holomorphic on an open set containing the disc $\bar{B}(a,r)$ then the values of f inside the disc are completely determined by the values of f on the boundary circle. Moreover, the formula can be interpreted as saying the value of $f(w)$ for w inside the circle is obtained as the "convolution" of f and the function $1/(z - w)$ on the boundary circle. Since the function $1/(z - w)$ is infinitely differentiable one can use this to show that f itself is infinitely differentiable as we will shortly show. If you take the Integral Transforms, you will see convolution play a crucial role in the theory of transforms. In particular, the convolution of two functions often inherits the "good" properties of either.

16.2. **Applications of the Integral Formula.** One immediate application of the Integral formula is known as Liouville's theorem, which will give an easy proof of the Fundamental Theorem of Algebra[41]. We say that a function $f\colon \mathbb{C} \to \mathbb{C}$ is *entire* if it is complex differentiable on the whole complex plane.

Theorem 16.15. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. If f is bounded then it is constant.*

Proof. Suppose that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $\gamma_R(t) = Re^{2\pi i t}$ be the circular path centred at the origin with radius R. The for $R > |w|$ the integral formula shows

$$\begin{aligned}
|f(w) - f(0)| &= \Big|\int_{\gamma_R} f(z)\Big(\frac{1}{z-w} - \frac{1}{z}\Big)dz\Big| \\
&= \Big|\int_{\gamma_R} \frac{w.f(z)}{z(z-w)}dz\Big| \\
&\leq 2\pi R \sup_{z:|z|=R} \Big|\frac{w.f(z)}{z(z-w)}\Big| \\
&\leq 2\pi R.\frac{M|w|}{R.(R-|w|)} = \frac{2\pi M|w|}{R-|w|},
\end{aligned}$$

Thus letting $R \to \infty$ we see that $|f(w) - f(0)| = 0$, so that f is constant an required.

□

Theorem 16.16. *Suppose that $p(z) = \sum_{k=0}^{n} a_k z^k$ is a non-constant polynomial where $a_k \in \mathbb{C}$ and $a_n \neq 0$. Then there is a $z_0 \in \mathbb{C}$ for which $p(z_0) = 0$.*

[41]Which, when it comes down to it, isn't really a theorem in algebra. The most "algebraic" proof of that I know uses Galois theory, which you can learn about in Part B.

Proof. By rescaling p we may assume that $a_n = 1$. If $p(z) \neq 0$ for all $z \in \mathbb{C}$ it follows that $f(z) = 1/p(z)$ is an entire function (since p is clearly entire). We claim that f is bounded. Indeed since it is continuous it is bounded on any disc $\bar{B}(0, R)$, so it suffices to show that $|f(z)| \to 0$ as $z \to \infty$, that is, to show that $|p(z)| \to \infty$ as $z \to \infty$. But we have

$$|p(z)| = |z^n + \sum_{k=0}^{n-1} a_k z^k| = |z^n|\{|1 + \sum_{k=0}^{n-1} \frac{a_k}{z^n - k}|\} \geq |z^n|.(1 - \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}}).$$

Since $\frac{1}{|z|^m} \to 0$ as $|z| \to \infty$ for any $m \geq 1$ it follows that for sufficiently large $|z|$, say $|z| \geq R$, we will have $1 - \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}} \geq 1/2$. Thus for $|z| \geq R$ we have $|p(z)| \geq \frac{1}{2}|z|^n$. Since $|z|^n$ clearly tends to infinity as $|z|$ does it follows $|p(z)| \to \infty$ as required. □

Remark 16.17. The crucial point of the above proof is that one term of the polynomial – the leading term in this case– dominates the behaviour of the polynomial for large values of z. All proofs of the fundamental theorem hinge on essentially this point. Note that $p(z_0) = 0$ if and only if $p(z) = (z - z_0)q(z)$ for a polynomial $q(z)$, thus by induction on degree we see that the theorem implies that a polynomial over $\mathbb{C}$ factors into a product of degree one polynomials.

Lemma 16.18. *Suppose that* $\gamma\colon [0, 1] \to \mathbb{C}$ *is a circular path,* $\gamma(t) = a + re^{2\pi it}$ *whose image bounds the disk* $B(a, r)$*. Then if* $g\colon \partial B(a, r) \to \mathbb{C}$ *is any continuous function, the function* $f\colon B(a, r) \to \mathbb{C}$ *defined by*

$$f(z) = \int_\gamma \frac{g(\zeta)}{\zeta - z} d\zeta$$

is given by a power series $\sum_{n\geq 0} c_n(z - a)^n$ *where we have*

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{g(\zeta)}{(\zeta - a)^{n+1}} d\zeta$$

Proof. Translating if necessary, we may assume that $a = 0$. Now if $z \in B(0,r)$ we have $|z| < |\zeta|$ for all ζ in the image of γ, hence we have $\frac{1}{\zeta - z} = \sum_{k=0}^{\infty} \frac{z^k}{\zeta^{k+1}}$, where the series converges absolutely for $|z| < |\zeta|$, and uniformly if we bound $|z| < K|\zeta|$ for some $K < 1$. Thus since the image of γ is compact and so $|g(z)|$ is bounded on it, we have $g(\zeta)/(\zeta - z)$ is the uniform limit $\sum_{k \geq 0} \frac{g(z)z^k}{\zeta^{k+1}}$ for all z in the image of γ. It follows from Lemma 15.16 that

$$2\pi i f(z) = \int_\gamma \frac{g(\zeta)}{\zeta - z} d\zeta = \int_\gamma \sum_{k \geq 0} \frac{g(\zeta)z^k}{\zeta^{k+1}} d\zeta = \sum_{k \geq 0} \Big(\int_\gamma \frac{g(\zeta)}{\zeta^{k+1}} dz \Big) z^k,$$

hence the claim follows.

□

This Lemma combined with the Integral Formula for holomorphic functions on an open set U has the very important consequence that any holomorphic function is both infinitely differentiable and equal to its Taylor series every point $a \in U$.

Theorem 16.19. *(Taylor expansions): Suppose that U is an open subset of $\mathbb{C}$ and $f\colon U \to \mathbb{C}$ is holomorphic on U. Then if $\bar{B}(a,r) \subset U$, the function f is given in $B(a,r)$by a power series $\sum_{n \geq 0} c_n(z-a)^n$ about a where*

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz$$

where $\gamma(t) = a + re^{2\pi i t}$. In particular, any holomorphic function is in fact infinitely complex differentiable.x

Proof. The fact that f is equal to a power series on $B(a,r)$ and the integral expression for the coefficients follows immediately from Lemma 16.18 since by Cauchy's integral formula we have for any $z \in B(a,r)$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

where $\gamma(t) = a + re^{2\pi it}$. (Since it is holomorphic on U it is certainly continuous on the image of γ.) The formulas for the coefficients of the power series in terms of the derivatives $f^{(n)}(a)$ follow from standard properties of power series. $\square$

Theorem 16.20. *(Cauchy's Integral Formulae for a circle): If $f: U \to \mathbb{C}$ is a holomorphic function on an open subset U of $\mathbb{C}$ and $\bar{B}(a,r) \subseteq U$ then for all $w \in B(a,r)$ we have*

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{n+1}} dz. \tag{16.1}$$

where $\gamma(t) = a + re^{it}$ is a parametrization of the boundary of $B(a,r)$.

Proof. First note that if $w \in B(a,r)$ then if $\delta = r - |w-a|$, we have $\bar{B}(w, \delta/2) \subseteq B(a,r)$ and since f is holomorphic in $B(a,r)$, applying Taylor's theorem to $\bar{B}(w,\delta/2)$ we see that $f(z) = \sum_{k=0}^\infty c_k(z-w)^k$, where $c_k = f^{(k)}(w)/k!$ in $B(w,\delta/2)$. Thus if we set $P_n(z)$ to be the polynomial $\sum_{k=0}^n c_k(z-w)^k$, it follows that $g(z) = (f(z) - P_n(z))/(z-w)^{n+1}$ is holomorphic in U, since it is evidently so for $z \neq w$ and it is equal to the power series $\sum_{k=0}^\infty c_{k+n+1}(z-w)^k$ in $B(w,\delta/2)$. Hence by Cauchy's theorem for the convex domain $B(a,\delta)$ we have $\int_\gamma g(z)dz = 0$. However $P_n(z)/(z-w)^{n+1} = \sum_{k=1}^{n+1} c_{n+1-k}(z-w)^{-k}$, and for $k \geq 2$ each of the functions $(z-w)^{-k}$ has an antiderivative on $\mathbb{C}\backslash\{w\}$ so that by the fundamental theorem of calculus their integral over γ is zero. It follows that

$$\frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^n}dz = \frac{n!}{2\pi i}\int_\gamma \frac{P(z)}{(z-w)^n}dz = \frac{n!}{2\pi i}\int_\gamma \frac{c_n}{z-w}dz = f^{(n)}(w)$$

where in the last equality we used Lemma 16.10. $\square$

Definition 16.21. A function which is locally given by a power series is said to be *analytic*. We have thus shown that any holomorphic function is actually analytic, and from now on we may use the terms interchangeably (as you may notice is common practice in many textbooks).

Corollary 16.22. *(Riemann's removable singularity theorem): Suppose that U is an open subset of $\mathbb{C}$ and $z_0 \in U$. If $f: U \backslash \{z_0\} \to \mathbb{C}$ is a holomorphic and bounded near z_0, then f extends to a holomorphic function on all of U.*

Proof. Fix $r > 0$ such that $\bar{B}(z_0, r) \subseteq U$. Then by the extension of Cauchy's integral formula given in Remark 16.13 we have for all $z \in B(z_0, r) \backslash \{z_0\}$

$$f(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $\gamma(t) = z_0 + re^{2\pi i t}$. Since by Lemma 16.18 the right-hand side defines a holomorphic function on all of $B(z_0, r)$ it defines the required extension. $\square$

We end this section with a kind of converse to Cauchy's theorem:

Theorem 16.23. *(Morera's theorem) Suppose that $f: U \to \mathbb{C}$ is a continuous function and on an open subset $U \subseteq \mathbb{C}$. If for any closed path $\gamma: [a, b] \to U$ we have $\int_\gamma f(z) dz = 0$, then f is holomorphic.*

Proof. By Theorem 15.21 we know that f has a primitive $F: U \to \mathbb{C}$. But then F is holomorphic on U and so infinitely differentiable on U, thus in particular $f = F'$ is also holomorphic. $\square$

Remark 16.24. One can prove variants of the above theorem: If U is a star-like domain for example, then our proof of Cauchy's theorem for such domains shows that $f: U \to \mathbb{C}$ has a primitive (and hence will be differentiable itself) provided $\int_T f(z) dz = 0$ for every triangle in U. In fact the assumption that $\int_T f(z) dz = 0$ for all triangles whose interior lies in U suffices to imply f is holomorphic for *any* open subset U: To show f is holomorphic on U, it suffices to show that f is holomorphic on $B(a, r)$ for each open disk $B(a,r) \subset U$. But this follows from the above as disks are star-like (in fact convex). It follows that we can characterize the fact that $f\colon U \to \mathbb{C}$ is holomorphic on U by an integral condition: $f\colon U \to \mathbb{C}$ is holomorphic if and only if for all triangles T which bound a solid triangle $\mathcal{T}$ with $\mathcal{T} \subset U$, the integral $\int_T f(z)dz = 0$.

This characterization of the property of being holomorphic has some important consequences:

Proposition 16.25. *Suppose that U is a domain and the sequence of functions $f_n\colon U \to \mathbb{C}$ converges to $f\colon U \to \mathbb{C}$ uniformly on every compact subset $K \subseteq U$. Then f is holomorphic.*

Proof. Since the property of being holomorphic is local, it suffices to show for each $w \in U$ that there is a ball $B(w,r) \subseteq U$ within which f is holomorphic. Since U is open, for any such w we may certainly find $r > 0$ such that $B(w,r) \subseteq U$. Then as $B(w,r)$ is convex, Cauchy's theorem for a star-like domain shows that for every closed path $\gamma\colon [a,b] \to B(w,r)$ whose image lies in $B(w,r)$ we have $\int_\gamma f_n(z)dz = 0$ for all $n \in \mathbb{N}$.

But $\gamma^* = \gamma([a,b])$ is a compact subset of U, hence $f_n \to f$ uniformly on γ^*. It follows that

$$0 = \int_\gamma f_n(z)dz \to \int_\gamma f(z)dz,$$

so that the integral of f around any closed path in $B(w,r)$ is zero. But then Theorem 15.21 shows that f has a primitive F on $B(w,r)$. But we have seen that any holomorphic function is in fact infinitely differentiable, so it follows that F, and hence f is infinitely differentiable on $B(w,r)$ as required.

□

Remark 16.26. The condition that $f_n \to f$ uniformly on any compact subset of U may seem strange at first sight, but it in fact the condition that is most often satisfied (and also the one the proof requires). A good example is to consider $f_n(z) = \sum_{k=0}^{n} z^k$. Then $f_n \to f$ where $f(z) = 1/(1-z)$ on $B(0,1)$, but the convergence is only uniform on the closed balls $\bar{B}(0,r)$ for $r < 1$, and *not*[42] on the whole of $B(0,1)$. You can check this is equivalent to the condition that f_n tends to f uniformly on any compact subset of $B(0,1)$.

Often functions on the complex plane are defined in terms of integrals. It is thus useful to have a criterion by which one can check if such a function is holomorphic. The following theorem gives such a criterion.

Theorem 16.27. *Let U be an open subset of $\mathbb{C}$ and suppose that $F \colon U \times [a,b]$ is a function satisfying*

(1) *The function $z \mapsto F(z,s)$ is holomorphic in z for each $s \in [a,b]$.*
(2) *F is continuous on $U \times [a,b]$*

Then the function $f \colon U \to \mathbb{C}$ defined by

$$f(z) = \int_a^b F(z,s)ds$$

is holomorphic.

Proof. Changing variables we may assume that $[a,b] = [0,1]$ (explicitly, one replaces s by $(s-a)/(b-a)$). By Theorem 16.25 it is enough to show that we may find a sequence of holomorphic functions $f_n(z)$ which converge of $f(z)$ uniformly on compact subsets of U. To find such a sequence, recall from Prelims Analysis that the Riemann integral of a continuous function is equal to the limit of its Riemann sums as the mesh of the partition used for the sum tends to zero. Using the partition $x_i = i/n$ for $0 \leq i \leq n$ evaluating

[42] If you have not already done it, then it is a good exercise to check that f_n does not converge uniformly to f on $B(0,1)$.

at the right-most end-point of each interval, we see that

$$f_n(z) = \frac{1}{n}\sum_{i=1}^{n} F(z, i/n),$$

is a Riemann sum for the integral $\int_0^1 F(z,s)ds$, hence as $n \to \infty$ we have $f_n(z) \to f(z)$ for each $z \in U$, *i.e.* the sequence (f_n) converges pointwise to f on all of U. To complete the proof of the theorem it thus suffices to check that $f_n \to f$ as $n \to \infty$ uniformly on compact subsets of U. But if $K \subseteq U$ is compact, then since F is clearly continuous on the compact set $K \times [0,1]$, it is uniformly continuous there, hence, given any $\epsilon > 0$, there is a $\delta > 0$ such that $|F(z,s) - F(z,t)| < \epsilon$ for all $z \in \bar{B}(a,\rho)$ and $s,t \in [0,1]$ with $|s-t| < \delta$. But then if $n > \delta^{-1}$ we have for all $z \in K$

$$\begin{aligned}
|f(z) - f_n(z)| &= \big|\int_0^1 F(z,s)dz - \frac{1}{n}\sum_{i=1}^{n} F(z,i/n)\big| \\
&= \left|\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n} \big(F(z,s) - F(z,i/n)\big)ds\right| \\
&\leq \sum_{i=1}^{n}\int_{(i-1)/n}^{i/n} |F(z,s) - F(z,i/n)|ds \\
&< \sum_{i=1}^{n} \epsilon/n = \epsilon.
\end{aligned}$$

Thus $f_n(z)$ tends to $f(z)$ uniformly on K as required. □

Example 16.28. If f is any continuous function on $[0,1]$, then the previous theorem shows that the function $f(z) = \int_0^1 e^{isz}f(s)ds$ is holomorphic in z, since clearly $F(z,s) = e^{isz}f(z)$ is continuous as a function on $\mathbb{C} \times [0,1]$ and, for fixed $s \in [0,1]$, F is holomorphic as a function of z. Integrals of this nature (though perhaps over the whole real line or the positive real axis) arise frequently in many parts of mathematics, as you can learn more about in the optional course on Integral Transforms.

Remark 16.29. Another way to prove the theorem is to use Morera's theorem directly: if $\gamma\colon [0,1] \to \mathbb{C}$ is a closed path in $B(a,r)$, then we have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma \big(\int_0^1 F(z,s)ds\big)dz \\ &= \int_0^1 \big(\int_\gamma F(z,s)dz\big)ds = 0,\end{aligned}$$

where in the first line we interchanged the order of integration, and in the second we used the fact that $F(z,s)$ is holomorphic in z and Cauchy's theorem for a disk. To make this completely rigorous however, one has to justify the interchange of the orders of integration. Next term's course on Integration proves a very general result of this form known as Fubini's theorem, but for continous functions on compact subets of $\mathbb{R}^n$ one can give more elementary arguments by showing any such function is a uniform limit of linear combinations of indicator functions of "boxes" – the higher dimensional analogues of step functions – and the elementary fact that the interchange of the order of integration for indicator functions of boxes holds trivially.

17. The identity theorem, isolated zeros and singularities

The fact that any complex differentiable function is in fact analytic has some very surprising consequences – the most striking of which is perhaps captured by the "Identity theorem". This says that if f, g are two holomorphic functions defined on a domain U and we let $S = \{z \in U : f(z) = g(z)\}$ be the locus on which they are equal, then if S has a limit point in U it must actually be all of U. Thus for example if there is a disk $B(a,r) \subseteq U$ on which f and g agree (not matter how small r is), then in fact they are equal on all of U! The key to the proof of the Identity theorem is the following result on the zeros of a holomorphic function:

Proposition 17.1. *Let U be an open set and suppose that $g\colon U \to \mathbb{C}$ is holomorphic on U. Let $S = \{z \in U : g(z) = 0\}$. If $z_0 \in S$ then either z_0 is isolated in S (so that g is non-zero in some disk about z_0 except at z_0 itself) or $g = 0$ on a neighbourhood of z_0. In the former case there is a unique integer $k > 0$ and holomorphic function g_1 such that $g(z) = (z - z_0)^k g_1(z)$ where $g_1(z_0) \neq 0$.*

Proof. Pick any $z_0 \in U$ with $g(z_0) = 0$. Since g is analytic at z_0, if we pick $r > 0$ such that $\bar{B}(a, r) \subseteq U$, then we may write

$$g(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k,$$

for all $z \in B(z_0, r) \subseteq U$, where the coeficients c_k are given as in Theorem 16.19. Now if $c_k = 0$ for all k, it follows that $g(z) = 0$ for all $z \in B(0, r)$. Otherwise, we set $k = \min\{n \in \mathbb{N} : c_n \neq 0\}$ (where since $g(z_0) = 0$ we have $c_0 = 0$ so that $k \geq 1$). Then if we let $g_1(z) = (z - z_0)^{-k} g(z)$, clearly $g_1(z)$ is holomorphic on $U \backslash \{z_0\}$, but since in $B(z_0, r)$ we have we have $g_1(z) = \sum_{n=0}^{\infty} c_{k+n}(z - z_0)^n$, it follows if we set $g_1(z_0) = c_k \neq 0$ then g_1 becomes a holomorphic function on all of U. Since g_1 is continuous at z_0 and $g_1(z_0) \neq 0$, there is an $\epsilon > 0$ such that $g_1(z) \neq 0$ for all $z \in B(z_0, \epsilon)$. But $(z - z_0)^k$ vanishes only at z_0, hence it follows that $g(z) = (z - z_0)^k g_1(z)$ is non-zero on $B(a, \epsilon) \backslash \{z_0\}$, so that z_0 is isolated.

Finally, to see that k is unique, suppose that $g(z) = (z - z_0)^k g_1(z) = (z - z_0)^l g_2(z)$ say with $g_1(z_0)$ and $g_2(z_0)$ both nonzero. If $k < l$ then $g(z)/(z - z_0)^k = (z - z_0)^{l-k} g_2(z)$ for all $z \neq z_0$, hence as $z \to z_0$ we have $g(z)/(z - z_0)^k \to 0$, which contradicts the assumption that $g_1(z) \neq 0$. By symmetry we also cannot have $k > l$ so $k = l$ as required. □

Remark 17.2. The integer k in the previous proposition is called the *multiplicity* of the zero of g at $z = z_0$ (or sometimes the *order of vanishing*).

Theorem 17.3. *(Identity theorem): Let U be a domain and suppose that f_1, f_1 are holomorphic functions defined on U. Then if $S = \{z \in U : f_1(z) = f_2(z)\}$ has a limit point in U, we must have $S = U$, that is $f_1(z) = f_2(z)$ for all $z \in U$.*

Proof. Let $g = f_1 - f_2$, so that $S = g^{-1}(\{0\})$. We must show that if S has a limit point then $S = U$. Since g is clearly holomorphic in U, by Proposition 17.1 we see that if $z_0 \in S$ then either z_0 is an isolated point of S or it lies in an open ball contained in S. It follows that $S = V \cup T$ where $T = \{z \in S : z \text{ is isolated}\}$ and $V = \text{int}(S)$ is open. But since g is continuous, $S = g^{-1}(\{0\})$ is closed in U, thus $V \cup T$ is closed, and so $\text{Cl}_U(V)$, the closure[43] of V in U, lies in $V \cup T$. However, by definition, no limit point of V can lie in T so that $\text{Cl}_U(V) = V$, and thus V is open and closed in U. Since U is connected, it follows that $V = \emptyset$ or $V = U$. In the former case, all the zeros of g are isolated so that $S' = T' = \emptyset$ and S has no limit points. In the latter case, $V = S = U$ as required.

□

Remark 17.4. The requirement in the theorem that S have a limit point *lying in* U is essential: If we take $U = \mathbb{C}\backslash\{0\}$ and $f_1 = \exp(1/z) - 1$ and $f_2 = 0$, then the set S is just the points where f_1 vanishes on U. Now the zeros of f_1 have a limit point at $0 \notin U$ since $f(1/(2\pi i n)) = 0$ for all $n \in \mathbb{N}$, but certainly f_1 is not identically zero on U!

We now wish to study singularities of holomorphic functions. The key result here is Riemann's removable singularity theorem, Corollary 16.22.

[43]I use the notation $\text{Cl}_U(V)$, as opposed to $\bar{V}$, to emphasize that I mean the closure of V in U, not in $\mathbb{C}$, that is, $\text{Cl}_U(V)$ is equal to the union of V with the limits points of V which lie in U.

Definition 17.5. If U is an open set in $\mathbb{C}$ and $z_0 \in U$, we say that a function $f\colon U\backslash\{z_0\} \to \mathbb{C}$ has an *isolated singularity* at z_0 if it is holomorphic on $B(z_0, r)\backslash\{z_0\}$ for some $r > 0$.

Suppose that z_0 is an isolated singularity of f. If f is bounded near z_0 we say that f has a *removable singularity* at z_0, since by Corollary 16.22 it can be extended to a holomorphic function at z_0. If f is not bounded near z_0, but the function $1/f(z)$ has a removable singularity at z_0, that is, $1/f(z)$ extends to a holomorphic function on all of $B(z_0, r)$, then we say that f has a *pole* at z_0. By Proposition 17.1 we may write $(1/f)(z) = (z - z_0)^m g(z)$ where $g(z_0) \neq 0$ and $m \in \mathbb{Z}_{>0}$. (Note that the extension of $1/f$ to z_0 must vanish there, as otherwise f would be bounded near z_0.) We say that m is the *order* of the pole of f at z_0. In this case we have $f(z) = (z-z_0)^{-m}.(1/g)$ near z_0, where $1/g$ is holomorphic near z_0 since $g(z_0) \neq 0$. If $m = 1$ we say that f has a *simple pole* at z_0.

Finally, if f has an isolated singularity at z_0 which is not removable nor a pole, we say that z_0 is an *essential singularity.*

Lemma 17.6. *Let f be a holomorphic function with a pole of order m at z_0. Then there is an $r > 0$ such that for all $z \in B(z_0, r)\backslash\{z_0\}$ we have*

$$f(z) = \sum_{n \geq -m} c_n (z - z_0)^n$$

Proof. As we have already seen, we may write $f(z) = (z - z_0)^{-m} h(z)$ where m is the order of the pole of f at z_0 and $h(z)$ is holomorphic and non-vanishing at z_0. The claim follows since, near z_0, $h(z)$ is equal to its Taylor series at z_0, and multiplying this by $(z - z_0)^{-m}$ gives a series of the required form for $f(z)$. □

Definition 17.7. The series $\sum_{n \geq -m} c_n (z - z_0)^n$ is called the *Laurent series* for f at z_0. We will show later that if f has an isolated essential singularity it still has a Laurent series expansion, but the series is then involves infinitely many positive and negative powers of $(z - z_0)$.

A function on an open set U which has only isolated singularities all of which are poles is called a *meromorphic* function on U. (Thus, strictly speaking, it is a function only defined on the complement of the poles in U.)

Lemma 17.8. *Suppose that f has an isolated singularity at a point z_0. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Proof. If z_0 is a pole of f then $1/f(z) = (z - z_0)^k g(z)$ where $g(z_0) \neq 0$ and $k > 0$. But then for $z \neq z_0$ we have $f(z) = (z - z_0)^{-k}(1/g(z))$, and since $g(z_0) \neq 0$, $1/g(z)$ is bounded away from 0 near z_0, while $|(z - z_0)^{-k}| \to \infty$ as $z \to z_0$, so $|f(z)| \to \infty$ as $z \to z_0$ as required.

On the other hand, if $|f(z)| \to \infty$ as $z \to z_0$, then $1/f(z) \to 0$ as $z \to z_0$, so that $1/f(z)$ has a removable singularity and f has a pole at z_0. □

Remark 17.9. The previous Lemma motivates the following definition: The *extended complex plane* $\mathbb{C}_\infty$ is the set $\mathbb{C} \cup \{\infty\}$ where ∞ is taken to be an additional point "at infinity". We will see later in the course that there is a natural way to make $\mathbb{C}_\infty$ into a metric space so that if $f \colon U \to \mathbb{C}$ is a meromorphic function on a domain U in $\mathbb{C}$, and we set $f(z_0) = \infty$ whenever f has a pole at z_0, then f becomes a continuous function from U to $\mathbb{C}_\infty$.

The case where f has an essential singularity is more complicated. We prove that near an isolated singularity the values of a holomorphic function are dense:

Theorem 17.10. *(Casorati-Weierstrass): Let U be an open subset of $\mathbb{C}$ and let $a \in U$. Suppose that $f \colon U \backslash \{a\} \to \mathbb{C}$ is a holomorphic function with an isolated essential singularity at a. Then for all $\rho > 0$ with $B(a, \rho) \subseteq U$,* *the set* $f(B(a,\rho)\backslash\{a\})$ *is dense in* $\mathbb{C}$*, that is, the closure of* $f(B(a,\rho)\backslash\{a\})$ *is all of* $\mathbb{C}$*.*

Proof. Suppose, for the sake of a contradiction, that there is some $\rho > 0$ such that $z_0 \in \mathbb{C}$ is not a limit point of $f(B(a,\rho)\backslash\{a\})$. Then the function $g(z) = 1/(f(z)-z_0)$ is bounded and non-vanishing on $B(a,\rho)\backslash\{a\}$, and hence by Riemann's removable singularity theorem, it extends to a holomorphic function on all of $B(a,\rho)$. But then $f(z) = z_0 + 1/g(z)$ has at most a pole at a which is a contradiction. $\square$

Remark 17.11. In fact much more is true: Picard showed that if f has an isolated essential singularity at z_0 then in any open disk about z_0 the function f takes every complex value infinitely often with at most one exception. The example of the function $f(z) = \exp(1/z)$, which has an essential singularity at $z = 0$ shows that this result is best possible, since $f(z) \neq 0$ for all $z \neq 0$.

17.1. **Principal parts.**

Definition 17.12. Recall that by Lemma 17.6 if a function f has a pole of order k at z_0 then near z_0 we may write

$$f(z) = \sum_{n \geq -k} c_n (z - z_0)^n.$$

The function $\sum_{n=-k}^{-1} c_n(z-z_0)^n$ is called the *principal part* of f at z_0, and we will denote it by $P_{z_0}(f)$. It is a rational function which is holomorphic on $\mathbb{C}\backslash\{z_0\}$. Note that $f - P_{z_0}(f)$ is holomorphic at z_0 (and also holomorphic wherever f is). The *residue* of f at z_0 is defined to be the coefficient c_{-1} and denoted $\mathrm{Res}_{z_0}(f)$.

The most important term in the principal part $P_{z_0}(f)$ is the term $c_{-1}/(z-z_0)$. This is because every other term has a primitive on $\mathbb{C}\backslash\{z_0\}$, hence by the Fundamental Theorem of Calculus it is the only part which contributes to the integral of f around a circle centered at z_0. Indeed if γ is a circular path about z_0 we have

$$\int_\gamma f(z)dz = \int_\gamma P_{z_0}(f) = \int_\gamma \frac{c_{-1}}{z-z_0}dz = 2\pi i c_{-1},$$

where the first equality holds by Cauchy's theorem for starlike domain, since $f - P_{z_0}(f)$ is holomorphic in the disk bounded by the image of γ. This is the key to what is called the "calculus of residues" which will will study in detail later.

Lemma 17.13. *Suppose that f has a pole of order m at z_0, then*

$$Res_{z_0}(f) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z))$$

Proof. Since f has a pole of order m at z_0 we have $f(z) = \sum_{n\geq -m} c_n(z-z_0)^n$ for z sufficiently close to z_0. Thus

$$(z-z_0)^m f(z) = c_{-m} + c_{-m+1}(z-z_0) + \ldots + c_{-1}(z-z_0)^{m-1} + \ldots$$

and the result follows from the formula for the derivatives of a power series. □

Remark 17.14. The last lemma is perhaps most useful in the case where the pole is simple, since in that case no derivatives need to be computed. In fact there is a special case which is worth emphasizing: Suppose that $f = g/h$ is a ratio of two holomorphic functions defined on a domain $U \subseteq \mathbb{C}$, where h is non-constant. Then f is meromorphic with poles at the zeros[44] of h. In particular, if h has a simple zero at z_0 and g is non-vanishing there, then f correspondingly has a simple pole at z_0. Since the zero of h is simple at z_0,

[44]Strictly speaking, the poles of f form a subset of the zeros of h, since if g also vanishes at a point z_0, then f may have a removable singularity at z_0.

we must have $h'(z_0) \neq 0$, and hence by the previous result

$$\mathrm{Res}_{z_0}(f) = \lim_{z \to z_0} \frac{g(z)(z-z_0)}{h(z)} = \lim_{z \to z_0} g(z). \lim_{z \to z_0} \frac{z-z_0}{h(z)-h(z_0)} = g(z_0)/h'(z_0)$$

where the last equality holds by standard Algebra of Limits results.