Metric spaces and complex analysis

Mathematical Institute, University of Oxford
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Problem Sheet 4

1. Find all the solutions of $\sin z = 2$ and $\exp(z) = 1 + \sqrt{3}i$.

2. Determine the power series and radius of convergence for each of the following functions $f(z)$ centred at $a \in \mathbb{C}$.

(i) $f(z) = (3+2z)^{-1}, \quad a = 0;$ (ii) $f(z) = (1-z)^{-2}, \quad a = 3;$ (iii) $f(z) = \sin z, \quad a = 1.$

3. $i)$ Use the Binomial Theorem for a general exponent to show that

$$(1-z)^{-1/2} = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^k}{2^{2k}}.$$

Hence evaluate the sums

$$\sum_{k=0}^{\infty} \frac{1}{2^{3k}} \binom{2k}{k}, \qquad \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{6k}} \binom{4k}{2k}.$$

$ii)$. Put the following power series into closed form. What is the radius of convergence in each case?

$$\sum_{n=0}^{\infty} n^2 z^n; \qquad \sum_{n=0}^{\infty} \frac{z^n}{(n+2)\, n!}.$$

4. If $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $S \subset \mathbb{N}$, we say that S is an arithmetic progression if there are integers a, d such that $S = \{a + nd : n \in \mathbb{Z}_{\geq 0}\}$. We call d the *step* of the arithmetic progression. Show that $\mathbb{N}$ cannot be partitioned into finitely many arithmetic progressions with distinct steps (excluding the trivial case of one progression with $a = d = 1$).
[*Hint: Consider the power series* $\sum_{n=0}^{\infty} z^n$.]

5. Suppose that $f(z) = \sum_{n \geq 0} a_n z^n$ is a power series with radius of convergence R. Show that f has a power series expansion about any $z_0 \in B(0, R)$.
[*Hint: Let* $z = (z_0 + (z - z_0))$ *and use the binomial theorem.*]

6. $i)$ Suppose that $l(z)$ is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$ and satisfies $\exp l(z) = z$. Show that

$$l(z) = L(z) + 2n\pi i$$

for some $n \in \mathbb{Z}$ where $L(z)$ is the holomorphic branch of $\log$ defined in lectures.

$ii)$ Show that there is no holomorphic function $\lambda(z)$ on $\mathbb{C}\backslash\{0\}$ such that $\exp \lambda(z) = z$.

$iii)$ There are unique holomorphic branches of $\log z$, $\sqrt{z}$ and $\sqrt[3]{z}$ on the cut plane $\mathbb{C}\backslash$ {negative imaginary axis} such that $\log 1 = 0$; $\sqrt{1} = 1$; $\sqrt[3]{1} = 1$. For these branches determine

$$\log(1+i), \qquad \sqrt{-1-i}, \qquad \sqrt[3]{-2}, \qquad \sqrt{1-i}.$$

$iv)$ Let C denote the logarithmic spiral given in polar coordinates by $r = 2e^{\theta}$. There is a unique holomorphic branch of $\log$ on $\mathbb{C}\backslash C$ such that $\log 1 = 0$. For this branch determine

$$\log i, \qquad \log 3, \qquad \log(-1), \qquad \log 1000, \qquad \log(-1000), \qquad \log 2000.$$

7. $i)$ Show that

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2i}$$

is 1–1 on $U = \{z \in \mathbb{C} : -\pi/2 < \operatorname{Re} z < \pi/2\}$.

ii) Fix $k \in (-\pi/2, \pi/2)$. Show that the image under $\sin$ of the line segment $\{z \in \mathbb{C} : \operatorname{Re} z = k\}$ is a branch of a hyperbola.

Deduce that the image of U under $\sin$ is $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$.

8 (*Optional*). *i*) Suppose that $(a_n)_{n=1}^N$ and $(b_n)_{n=1}^N$ are two finite sequences of complex numbers. Write $B_N = \sum_{n=1}^N b_n$ (taking by convention $B_0 = 0$). Show that for any $1 \le M \le N$

$$\sum_{n=M}^{N} a_n b_n = (a_N B_N - a_M B_{M-1}) - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

ii) Now consider the power series $s(z) = \sum_{n=0}^{\infty} \frac{z^n}{n}$. Show that s has radius of convergence 1 and converges on $\{z \in \mathbb{C} : |z| = 1|\}\setminus\{1\}$.

iii) Show that, given any finite subset T of the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, there is a power series with radius of convergence 1 which converges on all of $S^1 \setminus T$.