Metric spaces and complex analysis

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Problem Sheet 5

Throughout this sheet, for $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ we let $\gamma(a, r)$ denote the positively oriented circle centred at a of radius $r > 0$.

1. *Green's Theorem* states, for a region D in the plane, bounded by (an) oriented closed curve(s)[1] C in $\mathbb{R}^2$ and for real-vaued L and M with continuous partial derivatives on D, then

$$\int_C (L\,\mathrm{d}x + M\,\mathrm{d}y) = \int\int_D (M_x - L_y)\,\mathrm{d}x\,\mathrm{d}y.$$

If we assume, for a holomorphic function $f = u+iv$, that u_x, u_y, v_x, v_y are continuous, show that Cauchy's Theorem follows from Green's Theorem, that is, show that for a function f which is holomorphic on the interior D of a closed curve C, we have $\int_C f(z)dz = 0$.
[*The terms "positively oriented" and "interior" should be interpreted as they were in multivariable calculus. We will discuss them more rigorously later in the course.*]

2. By making the substitution $z = re^{i\theta}$, and making clear any special cases, for each integer k determine $\int_{\gamma(0,r)} z^k\,\mathrm{d}z$ (where as usual $\gamma(0, r)$ is the path $\gamma(0, r)(t) = re^{it}$ for $t \in [0, 2\pi]$). By writing $\sin\theta = \left(e^{i\theta} - e^{-i\theta}\right)/2i$ rewrite the integral on the left as a path integral around $\gamma(0, 1)$ and deduce that

$$\int_0^{2\pi} \sin^{2n}\theta\,\mathrm{d}\theta = \frac{2\pi}{4^n}\binom{2n}{n}.$$

3. Let

$$f(z) = \frac{5z^2 - 8}{z^3 - 2z^2}.$$

Determine $\int_{\gamma(0,1)} f(z)\,\mathrm{d}z$. Describe different closed paths γ in $\mathbb{C}$ such that $\int_\gamma f(z)\,\mathrm{d}z$ equals

$$14\pi i, \qquad 18\pi i, \qquad -2\pi i.$$

4. Let

$$I = \int_{\gamma(0,1)} \frac{\operatorname{Re} z}{2z - i}\,\mathrm{d}z \qquad \text{and} \qquad J = \int_0^{2\pi} \frac{\cos^2\theta}{5 - 4\sin\theta}\,\mathrm{d}\theta.$$

Using only Cauchy's Integral Formula, evaluate I. (Take note that $\operatorname{Re} z$ is not holomorphic, to remedy this, try to take advantage of properties of the contour you are integrating over.)

By setting $z = e^{i\theta}$ in the integral for I, determine J.

5. Use the Fundamental Theorem of Calculus to show, for $|a| > r > 0$, that

$$\int_{\gamma(0,r)} \frac{\mathrm{d}z}{z - a} = 0.$$

By integrating $(R + z)/(z(R - z))$ around $\gamma(0, r)$ show, for $0 \leqslant r < R$, that

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{R^2 - 2Rr\cos\theta + r^2} = \frac{2\pi}{R^2 - r^2}.$$

6. Suppose that D is a domain bounded by a contour C, which we assume can be parameterized by a function $\gamma_1 : [0, 1] \to \mathbb{C}$ (that is, $C = \gamma_1^*$). Let $z_0 \in D$ and let $r > 0$ be small enough so that $\bar{B}(z_0, r) \subset D$. The region $D\backslash\bar{B}(z_0, r)$ is thus bounded by $C \cup \partial B(z_0, r)$. Use the result of question 1 to show that if f is holomorphic on $D\backslash\{z_0\}$ then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz,$$

where $\gamma_2(t) = z_0 + re^{it}$, $(0 \le t \le 2\pi)$.

Use this and question 2 to calculate

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2(t) + b^2\sin^2(t)}.$$

[1] Note that the boundary of a region in the plane, for example "with holes", may be a disjoint union of closed curves.

7. Let f be holomorphic on $\mathbb{C}$. Write down an integral expression for $f^{(n)}(0)$.

(i) Suppose that f is holomorphic on $\mathbb{C}$ and that there exist $M, R > 0$ and k a non-negative integer such that

$$|f(z)| \leqslant M\,|z|^k \qquad \text{for } |z| > R.$$

Prove that $f(z)$ is a polynomial of degree at most k.

(ii) What holomorphic functions f satisfy $|f(z)| \leqslant |z|^k$ for all $z \in \mathbb{C}$?

(iii) Let $p(z)$ be a polynomial. What holomorphic functions f satisfy $|f(z)| \leqslant |p(z)|$ for all $z \in \mathbb{C}$?

8. Show that the function $f(z) = z/\left(z^2 - 4z + 1\right)^2$ is holomorphic except at α and β in $\mathbb{C}$ such that $|\alpha| < 1 < |\beta|$. Show that the Taylor coefficient c_n of the function $g(z) = z/(z-\beta)^2$ *centred at* α equals

$$\frac{\alpha + n\beta}{(\beta - \alpha)^{n+2}}.$$

With reference only to Taylor's Theorem, evaluate

$$\int_{\gamma(0,1)} \frac{z\ \mathrm{d}z}{\left(z^2 - 4z + 1\right)^2}$$

and hence show that

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{(2 - \cos\theta)^2} = \frac{4\pi}{3\sqrt{3}}.$$