**Metric spaces and complex analysis**
Mathematical Institute, University of Oxford
Michaelmas Term 2018

**Problem Sheet 6**

1. *i)* Suppose that $f$ is holomorphic on $\mathbb{C}$ and that $\operatorname{Re} f(z) \geqslant 0$ for all $z$. Show that $f$ is constant. [Hint: consider $\exp(-f(z))$.]

*ii)* Let $f : \mathbb{C} \to \mathbb{C}$ be a doubly-periodic holomorphic function: that is there exist non-zero complex numbers $w_1$ and $w_2$ such that $w_1/w_2$ is not real and such that

$$f(z + w_1) = f(z + w_2) = f(z) \qquad \text{for all } z.$$

Show that $f$ is constant.

2. *i)* Suppose that a power series $\sum_{n=0}^{\infty} a_n z^n$ converges on all of $\mathbb{C}$ so that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function. Show that the power series $f(z)$ converges *uniformly* on all of $\mathbb{C}$ if and only if $f$ is a polynomial.

*ii)* Show that

$$\sum_{r=1}^{\infty} \frac{1}{(r-z)^2}$$

is holomorphic on $\mathbb{C}\backslash\mathbb{N}$. [Hint: the series' convergence is not uniform on all of $\mathbb{C}\backslash\mathbb{N}$ but, for any $z \in \mathbb{C}\backslash\mathbb{N}$, is uniform on some neighbourhood of $z$.]

3. State the *Identity Theorem.*

*i)* Let $(\alpha_n)$ be a sequence of complex numbers such that

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{k^n} = 0$$

for all $k = 1, 2, 3...$ Prove that $\alpha_n = 0$ for all $n$.

*ii)* Let $z_n$ be a sequence of distinct points in $D(0,1)$ such that $z_n \to 0$ and let $f : D(0,1) \to \mathbb{C}$ be holomorphic. Show that if $f(z_n) = \sin z_n$ for all $n$, then $f(z) = \sin z$ for all $z \in D(0,1)$.

*iii)* Let $z_n = 1 - \frac{1}{n}$. Find a function $f : D(0,1) \to \mathbb{C}$ which is holomorphic on $D(0,1)$ such that $f(z_n) = \sin z_n$ but such that $f$ is not equal to $\sin(z)$.

4. Identify the singularities of the following functions. Classify any singularities which are isolated.

$$\frac{1}{e^z - 1}, \qquad \frac{\sin 2\pi z}{z^3(2z-1)}, \qquad \sin\left(\frac{1}{z}\right), \qquad \overline{z}, \qquad \frac{1}{\exp\left(\frac{1}{z}\right) + 2}.$$

5. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is an entire function, and for each $z_0 \in \mathbb{C}$ the power series expansion $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ has at least one $c_n$ equals zero. Prove that $f(z)$ is a polynomial.
[*Hint: Note that* $n! c_n = f^{(n)}(z_0)$ *and that* $\mathbb{C}$ *is uncountable.*]

6. Let the function $f$ be defined and holomorphic in the punctured disc $D'(a; r)$ and let $\{c_n\}$ be the Laurent coefficients of $f$ at $a$. Assume that $f$ is bounded on $D'(a; r)$.

*i)* Use the Estimation Theorem to show that $c_n = 0$ for $n < 0$.
Deduce that it is possible to define $f(a)$ so that $f$, thus extended, is holomorphic at $a$.

*ii)* Let $g$ be a holomorphic function on $\mathbb{C}$ and assume there exists a finite constant $M$ such that

$$|g(z)| \leqslant M\,|\sin z| \quad (z \in \mathbb{C}).$$

Prove that $g(z) = K \sin z$ on $\mathbb{C}$, for some constant $K$.

7. Classify the singularities of the following functions

$$\text{(i)}\ \ \frac{\pi}{\tan \pi z} \qquad \text{(ii)}\ \ \frac{z^2 - z}{1 - \sin z} \qquad \text{(iii)}\ \ \frac{(\cos z - 1)}{(e^z - 1)^2}.$$

Calculate the residue at each of their singularities.

8. Let

$$F(z) = \frac{1}{(z-1)^2(z+2)}.$$

Find Laurent expansions for $F$ in

$$A_1 = D(0,1), \qquad A_2 = \{z : 1 < |z| < 2\}; \qquad A_3 = \left\{z : \sqrt{2} < |z - i| < \sqrt{5}\right\}.$$