Metric spaces and complex analysis
Mathematical Institute, University of Oxford
Michaelmas Term 2018

Problem Sheet 7[1]

1. Prove, for $a > 0$, that
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4 + a^4} = \frac{\pi}{a^3\sqrt{2}}.$$

2. Let $-1 < a < 1$. Show that
$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}.$$

3. Show that
$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{2x - 1}\,\mathrm{d}x = -\frac{\pi}{2}.$$

4. By considering the integral
$$\int_{\Gamma_n} \frac{\pi \mathrm{d}w}{w^2 \sin \pi w}$$
where Γ_n is the square in $\mathbb{C}$ with vertices $\pm(n + 1/2)(1 \pm i)$ show that
$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$
(You may assume that there exists C such that $|\csc \pi w| \leqslant C$ on Γ_n for all n and all w.)

5. Write down a definition of a branch of $\log(z + i)$ which is holomorphic in the cut-plane
$$\mathbb{C}\backslash\{z : \operatorname{Re} z = 0,\ \operatorname{Im} z \leqslant -1\}.$$
By integrating $\log(z + i)/(z^2 + 1)$ around a suitable closed path, evaluate
$$\int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+1}\,\mathrm{d}x$$
and, by taking real parts, show that
$$\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1}\,\mathrm{d}x = 2\pi \log 2.$$

6. Show that
$$\int_0^{\infty} \frac{\sin px \sin qx}{x^2}\,\mathrm{d}x = \frac{\pi \min(p, q)}{2},$$
where $p, q > 0$.

7. Let $a \in \mathbb{C}$ with $-1 < \operatorname{Re} a < 1$. By considering a rectangular contour with corners at R, $R + i\pi$, $-R + i\pi$, $-R$, show that
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}\,\mathrm{d}x = \pi \sec\left(\frac{\pi a}{2}\right)$$
and hence evaluate, for real n,
$$\int_{-\infty}^{\infty} \frac{\cos nx}{\cosh x}\,\mathrm{d}x.$$

8. (Optional) By considering the function $\exp(z - z^{-1})$, or otherwise, show that
$$\int_0^{2\pi} \cos(n\theta - 2\sin\theta)\,\mathrm{d}\theta = 2\pi \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!}.$$

[1] Questions due to Richard Earl.