B4.2 Functional Analysis II
Consultation Session 1

Luc Nguyen
luc.nguyen@maths

University of Oxford

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Paper 2013–Q3

Let H and K be Hilbert spaces, $T \in \mathscr{B}(H, K)$.

(a) Definition of adjoint operator T^*. Show $\|T\| = \|T^*\| = \|T^*T\|^{1/2}$, $(T^*T)^* = T^*T$ and $\mathrm{Ker}\, T = \mathrm{Ker}\, T^*T$. Show for a projection P on H,

$$P = P^* \Rightarrow \mathrm{Ker}\, P = (\mathrm{Im}\, P)^\perp \Rightarrow \|P\| \leq 1.$$

(b) Definition of T being an isometry. Show that if this is the case then $\langle Tx, Tw\rangle = \langle x, w\rangle$ for all $x, w \in H$.

(c) T is said to be a partial isometry if there exist closed subspaces H_1 and K_1 of H and K such that T maps H_1 isometrically into K_1 while $T = 0$ on $H_1^\perp$. Show that if T is a partial isometry, then so is T^* and that both T^*T and TT^* are projections. Conversely, show that if T^*T is a projection, then T is a partial isometry.

Paper 2013–Q3(a)(i)-(iii)

Let H and K be Hilbert spaces, $T \in \mathscr{B}(H, K)$. Define T^*. Show $\|T\| = \|T^*\| = \|T^*T\|^{1/2}$, $(T^*T)^* = T^*T$ and $\mathrm{Ker}\, T = \mathrm{Ker}\, T^*T$.

- I'll not go over bookwork. But I'd like to point out that the underlying identity that defines T^* is $\langle Tx, y\rangle = \langle x, T^*y\rangle$. The reason for the existence and uniqueness of T^* is a consequence of the Riesz representation theorem. It also gives $\|T\| = \|T^*\|$. In practice, one rearranges $\langle Tx, y\rangle$ as the inner product of x with something else, and read off T^*.
- For $x \in H$, we have
  $\|T^*Tx\| \leq \|T^*\|\|Tx\| \leq \|T^*\|\|T\|\|x\| = \|T\|^2\|x\|$ and so
  $\|T^*T\| \leq \|T\|^2$.
  Conversely

  $$\|Tx\|^2 = \langle Tx, Tx\rangle = |\langle x, T^*Tx\rangle| \leq \|x\|\|T^*Tx\| \leq \|T^*T\|\|x\|^2.$$

  This gives $\|Tx\| \leq \|T^*T\|^{1/2}\|x\|$ and so $\|T\| \leq \|T^*T\|^{1/2}$. We thus have $\|T\| = \|T^*T\|^{1/2}$.

Paper 2013–Q3(a)(i)-(iii)

Let H and K be Hilbert spaces, $T \in \mathscr{B}(H, K)$. Define T^*. Show $\|T\| = \|T^*\| = \|T^*T\|^{1/2}$, $(T^*T)^* = T^*T$ and $\mathrm{Ker}\, T = \mathrm{Ker}\, T^*T$.

- The proof of $(T^*T)^* = T^*T$ is routine.
- Let us now show that $\mathrm{Ker}\, T = \mathrm{Ker}\, T^*T$.
  Clearly if $Tx = 0$ then $T^*Tx = 0$ and so $\mathrm{Ker}\, T \subset \mathrm{Ker}\, T^*T$.
  Conversely, suppose $T^*Tx = 0$. Then

$$0 = \langle x, T^*Tx\rangle = \langle Tx, Tx\rangle = \|Tx\|^2,$$

  which implies $Tx = 0$. This proves $\mathrm{Ker}\, T \supset \mathrm{Ker}\, T^*T$, whence $\mathrm{Ker}\, T = \mathrm{Ker}\, T^*T$.

Paper 2013–Q3(a)(iv)

Let H be a Hilbert space. Show for a projection P on H,

$$P = P^* \Rightarrow \operatorname{Ker} P = (\operatorname{Im} P)^\perp \Rightarrow \|P\| \leq 1.$$

- Recall that $P \in \mathscr{B}(H)$ is a projection if $P^2 = P$.
- We know that for every bounded linear operator P,

  $$\operatorname{Ker} P = (\operatorname{Im} P^*)^\perp \quad \text{and} \quad \overline{\operatorname{Im} P} = (\operatorname{Ker} P^*)^\perp.$$

  (Proof?)
  The first stated deduction is clear from the above.
- Suppose now that $\operatorname{Ker} P = (\operatorname{Im} P)^\perp$ and we would like to show that $\|P\| \leq 1$, i.e. $\|Px\| \leq \|x\|$ for all $x \in H$.

Paper 2013–Q3(a)(iv)

Let H be a Hilbert space. Show for a projection P on H,

$$P = P^* \Rightarrow \operatorname{Ker} P = (\operatorname{Im} P)^\perp \Rightarrow \|P\| \le 1.$$

- How may we use $\operatorname{Ker} P = (\operatorname{Im} P)^\perp$? By the projection theorem, this implies that every $x \in H$ is written uniquely as

  $$x = y + z \text{ where } Py = 0 \text{ and } z \in \overline{\operatorname{Im} P}.$$

  (Note the closure.)
- Claim: $\operatorname{Im} P$ is closed. To see this observe that $P^2 = P$ implies that $\operatorname{Im} P = \operatorname{Ker}(I - P)$ which is closed.
- Hence every $x \in H$ is written uniquely as

  $$x = y + z \text{ where } Py = 0 \text{ and } z \in \operatorname{Im} P.$$

  Applying P to both side, we have $Px = Pz = z$ (since $z = Pw$ for some w and $P^2 = P$).

Paper 2013–Q3(a)(iv)

Let H be a Hilbert space. Show for a projection P on H,

$$P = P^* \Rightarrow \mathrm{Ker}\, P = (\mathrm{Im}\, P)^\perp \Rightarrow \|P\| \leq 1.$$

- Putting things together, we see that every x is written as

  $$x = y + z = y + Px \text{ where } Py = 0$$

  where $y \perp z = Px$.
- By Pythagoras' theorem, we have

  $$\|Px\|^2 \leq \|Px\|^2 + \|y\|^2 = \|x\|^2,$$

  i.e. $\|P\| \leq 1$.
  (Note we have actually proved that P is the orthogonal projection onto $\mathrm{Im}\, P$ (which is closed).)

Paper 2013–Q3(b)

Let H and K be Hilbert spaces, $T \in \mathscr{B}(H, K)$.
Show that if T is an isometry then $\langle Tx, Tw\rangle = \langle x, w\rangle$ for all $x, w \in H$.

- This uses a standard polarisation argument.
- We start with $\|T(x + w)\|^2 = \|x + w\|^2$, $\|Tx\| = \|x\|$ and $\|Tw\| = \|w\|$. Expanding gives

$$\langle Tx, Tw\rangle + \langle Tw, Tx\rangle = \langle x, w\rangle + \langle w, x\rangle,$$

  which means $\mathrm{Re}\,\langle Tx, Tw\rangle = \mathrm{Re}\,\langle x, w\rangle$.
- If the field is real, we are done. If the field is complex, we apply the above to ix and w to get that $\mathrm{Re}\,\langle T(ix), Tw\rangle = \mathrm{Re}\,\langle ix, w\rangle$. This means $\mathrm{Im}\,\langle Tx, Tw\rangle = \mathrm{Im}\,\langle x, w\rangle$ and so we are done.

Paper 2013–Q3(b)(i)

Let H and K be Hilbert spaces, $T \in \mathscr{B}(H, K)$.
Show that if T is a partial isometry, then so is T^* and that both T^*T and TT^* are projections.

- By definition, there exist closed subspaces $H_1 \subset H$ and $K_1 \subset K$ such that $T|_{H_1} : H_1 \to K_1$ is a surjective isometry and $T|_{H_1^\perp} = 0$.
- Claim $T^*|_{K_1} : K_1 \to H_1$ is a surjective isometry and $T^*|_{K_1^\perp} = 0$. This gives that T^* is a partial isometry.
- Indeed, we have $\mathrm{Ker}\, T^* = (\mathrm{Im}\, T)^\perp \stackrel{\text{why?}}{=} K_1^\perp$. This implies that $T^*|_{K_1^\perp} = 0$.
- We also have $\overline{\mathrm{Im}\, T^*} = (\mathrm{Ker}\, T)^\perp \stackrel{\text{why?}}{=} (H_1^\perp)^\perp = H_1$ (as H_1 is closed). In particular $\mathrm{Im}\,(T^*|_{K_1}) = \mathrm{Im}\, T^*$ is a dense subset of H_1.

Paper 2013–Q3(b)(i)

Let H and K be Hilbert spaces, $T \in \mathscr{B}(H, K)$.
Show that if T is a partial isometry, then so is T^* and that both T^*T and TT^* are projections.

- Now suppose $k \in K_1$, we have $k = Th$ for some $h \in H_1$ and

  $$\|T^*k\|^2 = \langle T^*k, T^*k\rangle = \langle TT^*k, k\rangle = \langle TT^*k, Th\rangle$$

  using that T is isometric, we can continue this identity:

  $$= \langle T^*k, h\rangle = \langle k, Th\rangle = \langle k, k\rangle = \|k\|^2.$$

  So $T^*|_{K_1} : K_1 \to H_1$ is isometric.
- To see that $Im(T^*|_{K_1}) = H_1$, we take $h \in H_1$ and aim to show that $h = T^*k$ for some $k \in K_1$. Let $(h_n) \subset \mathrm{Im}(T^*|_{K_1})$ be such that $h_n \to h$ (note $\mathrm{Im}(T^*|_{K_1})$ is dense in H_1). Write $h_n = T^*k_n$ with $k_n \in K_1$, then $\|k_n - k_m\| = \|h_n - h_m\| \to 0$ and so (k_n) is Cauchy, hence convergent to some $k \in K_1$ (since K_1 is closed). By continuity $h = T^*k$.

Paper 2013–Q3(b)(i)

Let H and K be Hilbert spaces, $T \in \mathscr{B}(H, K)$.
Show that if T is a partial isometry, then so is T^* and that both T^*T and TT^* are projections.

- For the last statement, it suffices to show T^*T is a projection. The other part is obtained by swapping the role of T and T^*.
- Take $x, y \in H$. We have $\langle T^*TT^*Tx, y\rangle = \langle TT^*Tx, Ty\rangle$. Write $y = a + b$ where $a \in H_1$ and $b \in H_1^\perp$ so that $Ty = Ta$. Then

$$\langle T^*TT^*Tx, y\rangle = \langle T \underbrace{T^*Tx}_{\in H_1}, T \underbrace{a}_{\in H_1}\rangle = \langle T^*Tx, a\rangle$$
$$= \langle \underbrace{T^*Tx}_{\in H_1}, a + \underbrace{b}_{\in H_1^\perp}\rangle = \langle T^*Tx, y\rangle.$$

Since x, y are arbitrary, this means $(T^*T)^2 = T^*T$ and so T^*T is a projection.

Paper 2013–Q3(b)(ii)

Let H and K be Hilbert spaces, $T \in \mathscr{B}(H, K)$.
Conversely, show that if T^*T is a projection then T is a partial isometry.

- We know that T^*T is self-adjoint. By (a), $\mathrm{Ker}\, T = \mathrm{Ker}\, T^*T$ and $\mathrm{Ker}\, T^*T = (\mathrm{Im}\, T^*T)^\perp$. In fact, we also know that $\mathrm{Im}\, T^*T$ is closed and T^*T is the orthogonal projection on to $\mathrm{Im}\, T^*T$.
- Let $H_1 = \mathrm{Im}\, T^*T$ so that $H_1^\perp = \mathrm{Ker}\, T$ (hence $T|_{H_1^\perp} = 0$).
- Claim: $T|_{H_1}$ is isometric. Let $h \in H_1$. Since T^*T is the orthogonal projection onto H_1, $h = T^*Th$. Hence

  $$\langle Th, Th \rangle = \langle T^*Th, h \rangle = \langle h, h \rangle = \|h\|^2,$$

  i.e. T is isometric.
- Finally, let $K_1 = \overline{\mathrm{Im}\, T|_{H_1}}$. Claim: $\mathrm{Im}\,(T|_{H_1})$ is actually K_1. This can be done as in the proof of $\mathrm{Im}\,(T^*|_{K_1}) = H_1$ which we did earlier on in (i).

Paper 2014 – Q3

(a) Bookwork.

(b) Let D denote the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in $\mathbb{C}$ and consider $L^2(D)$ with area measure. Let $A^2(D)$ be the set of functions $f : D \to \mathbb{C}$ such that f is holomorphic and $|f|^2$ is integrable. We identify with a subspace of $L^2(D)$. You are given that $A^2(D)$ is closed in $L^2(D)$.

Let $e_n = \left(\frac{n+1}{\pi}\right)^{1/2} z^n$, $n = 0, 1, \ldots$

(i) Prove that $(e_n)_{n\geq 0}$ is a complete orthonormal sequence in $A^2(D)$.

(ii) Prove that if $\sum |a_k|^2$ converges then the function $\sum_{k=0}^{\infty}(k+1)^{1/2} a_k z^k$ is holomorphic in D. Is the converse true?

Paper 2014 – Q3(b)(i)

Prove that $e_n = \left(\frac{n+1}{\pi}\right)^{1/2} z^n$, $n = 0, 1, \ldots$ form a complete orthonormal sequence in $A^2(D)$.

- It's straightforward to check that (e_n) is an orthonormal sequence in $A^2(D)$.
- To show that it's complete, suppose that $f \in A^2(D)$ with $\langle f, e_n \rangle = 0$ for all e_n, and we need to show that $f \equiv 0$.
- We know that f has a Taylor series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

  which converges uniformly on any disk $D(0, R) \subset D$ with $R < 1$ and $f \equiv 0$ if and only if $a_k = 0$ for all k.

Paper 2014 – Q3(b)(i)

- Let us compute $\langle f, e_n \rangle$.
- Since $f \in L^2(D)$, we have by the dominated convergence theorem that $f - f|_{D(0,R)} \to 0$ as $R \to 1$. It follows that

$$\langle f, e_n \rangle = \left(\frac{n+1}{\pi}\right)^{1/2} \lim_{R \to 1} \int_{D(0,R)} f(z)\bar{z}^n \, dA.$$

Using the uniform convergence of the Taylor series we then have

$$\begin{aligned}\langle f, e_n \rangle &= \left(\frac{n+1}{\pi}\right)^{1/2} \lim_{R \to 1} \sum_{k=0}^{\infty} \int_{D(0,R)} a_k z^k \bar{z}^n \, dA \\ &= \left(\frac{n+1}{\pi}\right)^{1/2} \lim_{R \to 1} \sum_{k=0}^{\infty} \int_0^R \int_0^{2\pi} a_k r^{k+n+1} e^{i(k-n)\theta} \, d\theta \, dr \\ &= \left(\frac{n+1}{\pi}\right)^{1/2} \lim_{R \to 1} \frac{\pi}{n+1} a_n R^{2n+2} = \left(\frac{\pi}{n+1}\right)^{1/2} a_n.\end{aligned}$$

- We deduce that $a_n = 0$ for all n and so $f \equiv 0$.

Paper 2014 – Q3(b)(i)

Prove that if $\sum |a_k|^2$ converges then the function $\sum_{k=0}^{\infty}(k+1)^{1/2} a_k z^k$ is holomorphic in D. Is the converse true?

- We have $\sum_{k=0}^{\infty}(k+1)^{1/2} a_k z^k = \frac{1}{\sqrt{\pi}} \sum a_k e_k$. Now if $\sum |a_k|^2$ converges, then $\sum a_k e_k$ belongs to the closed linear span of (e_n), i.e. $A^2(D)$, and hence is holomorphic.
- The converse doesn't hold: If f is holomorphic in D, it is not necessary that f can be written in the form $f = \sum_{k=0}^{\infty}(k+1)^{1/2} a_k z^k$ with $\sum |a_k|^2 < \infty$, as this latter means that $f \in A^2(D)$.
  To confirm this, we only need to exhibit a function f which is holomorphic in D but is not square integrable in D. We can take for example $f(z) = (1-z)^{-1}$. (This corresponds to $a_k = (k+1)^{-1/2}$ which is clearly not square summable.)

Paper 2013–Q2

(a) Projection theorem + Pythagoras' theorem.

(b) Let H be a Hilbert space, $C \subset K \subset H$ be non-empty closed convex subsets and P_C and P_K be the projections to those convex sets. Show that, if K is a subspace, then $P_C = P_C \circ P_K$, but this need not hold otherwise.

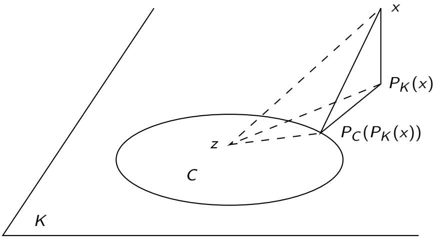
(c) Let $H = L^2(\mathbb{R})$ and

$$K = \{g \in H : g(t) = 0 \text{ for almost all } t \in (-\infty, 0)\},$$
$$C = \{g \in K : g(t) \geq 0 \text{ for almost all } t \in \mathbb{R}\},$$

Let f be a real-valued function in H. Find $P_K(f)$ and $P_C(f)$.

Paper 2013–Q2(b)

K is a closed subspace.



- Fix $x \in H$. We need to show that $P_C(P_K(x)) = P_C(x)$, i.e.

$$\|x - z\| \geq \|x - P_C(P_K(x))\| \quad (1)$$

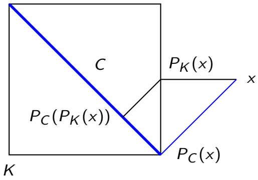
for all $z \in C$.

- Let $y = P_K(x)$. By Pythagoras' theorem, $\|x - z\|^2 = \|x - y\|^2 + \|y - z\|^2$.
- By definition of P_C, $\|y - z\| \geq \|y - \|P_C(y)\|$.
- So, $\|x - z\|^2 = \|x - y\|^2 + \|y - z\|^2 \geq \|x - y\|^2 + \|y - P_C(y)\|^2 = \|x - P_C(y)\|^2$, where we have used Pythagoras' theorem once more time. This gives (1).

Paper 2013–Q2(b)

Example of K (not a subspace) and C for which $P_C \neq P_C \circ P_K$.
Many such examples e.g.

$$K = [0,1]^2 \subset \mathbb{R}^2, \qquad C = \{(a,b) : a+b = 1, 0 \leq a, b \leq 1\}.$$



Paper 2013–Q2(c)

Let $H = L^2(\mathbb{R})$ and

$$K = \{g \in H : g(t) = 0 \text{ for almost all } t \in (-\infty, 0)\},$$
$$C = \{g \in K : g(t) \geq 0 \text{ for almost all } t \in \mathbb{R}\},$$

Let f be a real-valued function in H. Find $P_K(f)$ and $P_C(f)$.

- H and K are clearly closed and convex. Hence P_C and P_K are well-defined.
- Let $g \in K$, then

$$\|f - g\|^2 = \int_{-\infty}^{0} |f(t)|^2 \, dt + \int_0^\infty |f(t) - g(t)|^2 \, dt.$$

  In order for this to be smallest, we need $g(t) = f(t)$ for all most all $t \geq 0$. This means

$$P_K(f) = f \chi_{[0,\infty)}.$$

Paper 2013–Q2(c)

...

$$C = \{g \in K : g(t) \geq 0 \text{ for almost all } t \in \mathbb{R}\},$$

... Find ... $P_C(f)$.

- Let $g \in C$, then

$$\|f - g\|^2 = \int_{-\infty}^{0} |f(t)|^2 \, dt + \int_0^\infty |f(t) - g(t)|^2 \, dt$$

- If we minimize $|f(t) - g(t)|^2$ for each t under the constraint that $g(t) \geq 0$, we get $g(t) = f^+(t)$. This gives

$$\|f - g\|^2 \geq \int_{-\infty}^{0} |f(t)|^2 \, dt + \int_0^\infty |f(t) - f^+(t)|^2 \, dt.$$

  From here we see that

$$P_C(f) = f^+ \chi_{[0,\infty)}.$$

Paper 2014–Q1(c)(d)

Let X be a complex Hilbert space, $T \in \mathscr{B}(X)$.

(c) Prove that if there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in X$, then T is injective and $\mathrm{Im}\, T$ is closed. Prove further that if T is self-adjoint, then T is invertible in $\mathscr{B}(X)$.

(d) Suppose that T is self-adjoint. Prove that $\mathrm{i}\, I + T$ has an inverse and that $(\mathrm{i}\, I + T)^{-1}(\mathrm{i}\, I - T)$ is unitary.

Paper 2014–Q1(c)

Let X be a complex Hilbert space, $T \in \mathscr{B}(X)$.
Prove that if there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in X$, then T is injective and $\mathrm{Im}\, T$ is closed. Prove further that if T is self-adjoint, then T is invertible in $\mathscr{B}(X)$.

- The injectivity is clear. We have seen the part about the closedness of $\mathrm{Im}\, T$ when T is isometric. The proof now is the same.
- Suppose $(y_n = Tx_n) \subset \mathrm{Im}\, T$ and $y_n \to y$ in X.
- Then $\|x_n - x_m\| \leq \delta^{-1}\|y_n - y_m\|$. So (x_n) is Cauchy, hence converges to some $x \in X$. Continuity gives $y = Tx$. This proves $\mathrm{Im}\, T$ is closed.

Paper 2014–Q1(c)

Let X be a complex Hilbert space, $T \in \mathscr{B}(X)$.
Prove that if there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in X$, then T is injective and $\mathrm{Im}\, T$ is closed. Prove further that if T is self-adjoint, then T is invertible in $\mathscr{B}(X)$.

- Suppose now T is self-adjoint. Then $0 = \mathrm{Ker}\, T = \mathrm{Ker}\, T^* = (\mathrm{Im}\, T)^\perp$ and so $\mathrm{Im}\, T$ is dense in X. Since $\mathrm{Im}\, T$ is closed, we have that $\mathrm{Im}\, T = X$, i.e. T is surjective.
- It follows that T is a bijection and has an inverse T^{-1}. It is clear that T^{-1} is linear. One also has $\|T^{-1}y\| \leq \delta^{-1}\|TT^{-1}y\| = \delta^{-1}\|y\|$ and so T^{-1} is bounded.
  Comment: There is no need to use inverse mapping theorem.

Paper 2014–Q1(d)

Let X be a complex Hilbert space, $T \in \mathscr{B}(X)$.
Suppose that T is self-adjoint. Prove that $\mathrm{i}\, I + T$ has an inverse and that $(\mathrm{i}\, I + T)^{-1}(\mathrm{i}\, I - T)$ is unitary.

- We attempt to use (c). We compute

$$\|(\mathrm{i}\, I + T)x\|^2 = \|x\|^2 + \|Tx\|^2 - 2\mathrm{i}\,\mathrm{Im}\,\langle x, Tx\rangle.$$

- Since T is self-adjoint, $\langle x, Tx\rangle = \langle Tx, x\rangle = \overline{\langle x, Tx\rangle}$ and so $\langle x, Tx\rangle$ is real. Thus

$$\|(\mathrm{i}\, I + T)x\|^2 = \|x\|^2 + \|Tx\|^2 \geq \|x\|^2.$$

- By (c), $\mathrm{i}\, I + T$ is invertible. (Note that, a brief argument is needed as $\mathrm{i}\, I + T$ is not self-adjoint.)

Paper 2014–Q1(d)

Let X be a complex Hilbert space, $T \in \mathscr{B}(X)$.
Suppose that T is self-adjoint. Prove that $\mathrm{i}\, I + T$ has an inverse and that $(\mathrm{i}\, I + T)^{-1}(\mathrm{i}\, I - T)$ is unitary.

- Let $U = (\mathrm{i}\, I + T)^{-1}(\mathrm{i}\, I - T)$. Applying what we just proved to $-T$, we have also that $\mathrm{i}\, I - T$ is invertible and hence U is invertible.
- To conclude, we show that U is isometric by showing $U^* U = I$. First,

$$U^* = (-\mathrm{i}\, I - T^*)(-\mathrm{i}\, I + T^*)^{-1} = (\mathrm{i}\, I + T)(\mathrm{i}\, I - T)^{-1}.$$

- Next, since $\mathrm{i}\, I + T$ and $\mathrm{i}\, I - T$ commute, so do their inverses. It follows that

$$\begin{aligned} U^* U &= (\mathrm{i}\, I + T)(\mathrm{i}\, I - T)^{-1}(\mathrm{i}\, I + T)^{-1}(\mathrm{i}\, I - T) \\ &= (\mathrm{i}\, I + T)(\mathrm{i}\, I + T)^{-1}(\mathrm{i}\, I - T)^{-1}(\mathrm{i}\, I - T) = I. \end{aligned}$$