**Metric spaces and complex analysis**
Mathematical Institute, University of Oxford
Michaelmas Term 2018

**Problem Sheet 8**

1. Evaluate, using a keyhole contour cut along the positive real axis, or otherwise,

$$\int_0^\infty \frac{x^{1/2}\log x}{(1+x)^2}\mathrm{d}x.$$

2. Let $n \geqslant 2$. By using the contour comprising $[0,R]$, the circular arc from $R$ to $Re^{2\pi i/n}$, and $\left[0, Re^{2\pi i/n}\right]$, show that

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^n} = \frac{\pi}{n}\csc\left(\frac{\pi}{n}\right).$$

3. Suppose that $f\colon U \to \mathbb{C}$ is a holomorphic function on an open set $U \subseteq \mathbb{C}$, and suppose $f'(a) \neq 0$ for some $a \in U$. Show that there is an $r > 0$ such that $f$ is injective on $B(a,r)$ and that its inverse $g$ is given on such a disk by

$$g(w) = \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)-w}dz.$$

where $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$.

4. In each of the following cases find a conformal mapping from the given region $G$ onto the open unit disc $D(0,1)$.

*i)* $G = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$,

*ii)* $G = \{z \in \mathbb{C} : z \neq 0 \text{ and } -\pi/4 < \arg z < \pi/4\}$,

*iii)* $G = \{z \in \mathbb{C} : |z-i| < \sqrt{2} \text{ and } |z+i| > \sqrt{2}\}$.

5. Let

$$A = \{z \in \mathbb{C} : |z-2| < 2 \quad\text{and}\quad |z-1| > 1\}; \qquad B = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < \pi\}.$$

Find the image of $A$ under the map $z \mapsto 1/z$ and the image of $B$ under the map $z \mapsto \exp(iz)$.

Let $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Given $a, b \in H$ find a conformal bijection $H \to H$ of the form $z \mapsto \lambda z + \mu$ which maps $a$ to $b$.

Deduce that for any two points $a, b \in A$ there is a conformal bijection $f : A \to A$ such that $f(a) = b$.

6. Let $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\} \subset \mathbb{C}_\infty$.

(1) Show that group $\Gamma$ of Mobius transformations $T$ for which $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ are exactly those of the form $T(z) = (az+b)/(cz+d)$ where $a, b, c, d$ can be chosen to be real.

(2) Calculate the group $\Gamma_1$ of Mobius tranformations which preserve $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ [*Hint: Why must* $\Gamma_1$ *be a subgroup of* $\Gamma$*?*]

7. Suppose that $f(z) = (az+b)/(cz+d)$ is a Mobius transformation. We have seen any Mobius transformation is conformal. Find the subgroup $I$ of Mob, the group of all Mobius transformations, which consists of those Mobius transformations which are isometries with respect to the standard metric on the Riemann sphere (*i.e.* the one induced from the Euclidean metric on $\mathbb{R}^3$).

[*Hint: In the lecture notes it is shown that the distance on the Riemann sphere corresponds to the distance function on* $\mathbb{P}^1$ *given by*

$$d(L_1, L_2) = 2\sqrt{1 - \frac{|\langle v, w\rangle|^2}{\|v\|^2\|w\|^2|}},$$

*where* $v \in L_1\backslash\{0\}$ *and* $w \in L_2\backslash\{0\}$*. A Mobius map corresponds to the action of a matrix* $A \in GL_2(\mathbb{C})$ *on lines in* $\mathbb{C}^2$*. Which* $2 \times 2$ *matrices automatically preserve the above expression for* $d$*?*]

8. Write down the solution $u(x, y)$ to the Dirichlet problem for the following region and boundary conditions:

$$U = \{x + iy : 0 \leqslant y \leqslant 1\}, \qquad u(x, 0) = 0, \qquad u(x, 1) = 1.$$

Hence, using appropriate conformal maps, solve the Dirichlet problem for the following regions and boundary conditions.

*i)* $U = \{z : r_1 \leqslant |z| \leqslant r_2\}$, $\quad u(z) = 0$ when $|z| = r_1$, $\quad u(z) = 1$ when $|z| = r_2$.
*ii)* $U = \{z : \operatorname{Im} z \geqslant 0\}$, $\quad u(x, 0) = 0$ when $x > 0$, $\quad u(x, 0) = 1$ when $x < 0$.
*iii)* $U = \{z : |z| \leqslant 1\}$, $\quad u(z) = 0$ when $|z| = 1$ and $\operatorname{Im} z < 0$, $\quad u(z) = 1$ when $|z| = 1$ and $\operatorname{Im} z > 0$.
*iv)* $U = \{z : \operatorname{Im} z \geqslant 0\}$, $\quad u(x, 0) = 0$ when $|x| > 1$, $\quad u(x, 0) = 1$ when $|x| < 1$.
*v)* $U = \{z : \operatorname{Im} z \geqslant 0, -1 \leqslant \operatorname{Re} z \leqslant 1\}$, $\quad u(1, y) = u(-1, y) = 0$ when $y > 0$, $\quad u(x, 0) = 1$ when $|x| < 1$.

9. (*Optional.*) Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a meromorphic function on the extended complex plane.

*i)* Show that either $f$ is constant or $f^{-1}(a)$ is finite for each $a \in \mathbb{P}^1$.

*ii)* In particular it follows that $f$ has finitely many poles, say $S = \{s_1, s_2, \ldots, s_k\}$. Let $P_i$ be the principal part of $f$ at $s_i$ (so that $P_i$ for $s_i \neq \infty$ is a rational function which has a pole at $s_i$, takes the value 0 at $\infty$, and is finite on $\mathbb{C}$, while the "principal part" at $\infty$ is a polynomial in $z$, the coordinate in the chart $U_0$, which vanishes at 0). Show that $f - \sum_{i=1}^{k} P_i(f)$ is constant. [*Hint*: If $g = f - \sum_{i=1}^{n} P_i(f)$ then $f(\mathbb{P}^1)$ is compact and so closed in $\mathbb{P}^1$, and does not contain $\infty$.]

[*Note that this gives an "analytic proof" that any rational function has an expansion into "partial fractions".*].