26. Appendix II: Power series

In this appendix we give a proof of the following Theorem, which was established in Prelims Analysis I.

Proposition 26.1. *Let $s(z) = \sum_{k\geq 0} a_k z^k$ be a power series, let S be the domain on which it converges, and let R be its radius of convergence. Then power series $t(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$ also has radius of convergence R and on $B(0, R)$ the power series s is complex differentiable with $s'(z) = t(z)$. In particular, it follows that a power series is infinitely complex differentiable within its radius of convergence.*

Proof. First note that the power series $\sum_{k=1}^{\infty} k a_k z^{k-1}$ clearly has the same radius of convergence as $\sum_{k=1}^{\infty} k a_k z^k$, and by Lemma 13.22 this has radius of convergence[57]

$$\limsup_k |k a_k|^{1/k} = \lim_k (k^{1/k}) \limsup_k |a_k|^{1/k} = \limsup |a_k|^{1/k} = R,$$

since $\lim_{k\to\infty} k^{1/k} = 1$. Thus $s(z) = \sum_{k=0}^{\infty} a_k z^k$ and $t(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$ have the same radius of convergence. To see that $s(z)$ is complex differentiable with derivative $t(z)$, consider the sequence of polynomials f_n in two complex variables:

$$f_n(z, w) = a_n(\sum_{i=0}^{n-1} z^i w^{n-1-i}), \quad (n \geq 1).$$

Fix $\rho < R$, then for (z, w) with $|z|, |w| \leq \rho$ we have

$$|f_n(z, w)| = \big|a_n \sum_{i=0}^{n-1} z^i w^{n-i}\big| \leq |a_n| \sum_{i=0}^{n-1} |z|^i |w|^{n-i} \leq |a_n| n \rho^{n-1}$$

It therefore follows from the Weierstrass M-test with[58] $M_n = |a_n| n \rho^{n-1}$ that the series $\sum_{n\geq 0} f_n(z, w)$ converges uniformly (and absolutely) on $\{(z, w) : |z|, |w| \leq \rho\}$ to a function $F(z, w)$. In particular, it follows that $F(z, w)$ is continuous. But since $\sum_{k=1}^{n} f_k(z, z) = \sum_{k=1}^{n} k a_k z^{k-1}$, it follows that $F(z, z) = t(z)$. On the other hand, for $z \neq w$ we have $\sum_{i=0}^{k-1} z^i w^{k-i} = \frac{z^k - w^k}{z - w}$, so that

$$F(z, w) = \sum_{k=0}^{\infty} a_k \frac{z^k - w^k}{z - w} = \frac{s(z) - s(w)}{z - w},$$

hence it follows by the continuity of F that if we fix z with $|z| < \rho$ then

$$\lim_{z\to w} \frac{s(z) - s(w)}{z - w} = F(z, z) = t(z).$$

Since $\rho < R$ was arbitrary, we see that $s(z)$ is differentiable on $B(0, R)$ with derivative $t(z)$.

Finally, since we have shown that any power series is differentiable within its radius of convergence and its derivative is again a power series with the same radius of convergence, it follows by induction that any power series is in fact infinitely differentiable within its radius of convergence. □

[57]This uses a standard property of lim sup which is proved for completeness in Lemma 24.3 in Appendix I.

[58]We know $\sum_{n\geq 0} M_n = |a_n| n \rho^{n-1}$ converges since $\rho < R$ and $t(z)$ has radius of convergence R.