INTEGRAL TRANSFORMS – SHEET 1

(Exercises on lectures in Weeks 1 and 2)

The Dirac Delta Function and Distributions. The Laplace Transform. Applications to ODEs.

1. Let $\phi(x)$ be a test function. Show that the following are test functions:
(i) $\phi(ax+b)$ where $a>0$ and $b\in\mathbb{R}$.
(ii) $f(x)\phi(x)$ where $f(x)$ is an arbitary smooth function.
(iii) $\phi^{(k)}(x)$ where $k\in\mathbb{N}$.

2 (i) Let $0<a<1$. Solve the boundary-value problem:

$$f''(x)=\delta(x-a), \qquad f(0)=f(1)=0.$$

(ii) Let $a>0$ and $k\in\mathbb{R}$. Solve directly the initial value problem

$$f''(x)-3f'(x)+2f(x)=k\,\delta(x-a) \qquad f(0)=f'(0)=1.$$

3. (*Kick Stop*) Consider a mass on a spring where the extension of the spring $x(t)$ satisfies

$$m\ddot{x}+kx=I\,\delta(t-T),$$

where m is the mass, and $k>0$ is the spring constant. Suppose initially $x(0)=a$ and $\dot{x}(0)=0$ and that at time $t=T$ an instantaneous impulse I is applied to the mass.

Obtain the motion of the mass for $t>0$, and find conditions on I and T such that the impulse completely stops the motion. Explain the result physically.

4. Show that, for $a\neq 0$,

$$\delta(ax)=\frac{1}{|a|}\delta(x).$$

[Hint: use the approximating functions δ_n from lectures.] What is $\delta(x^2-a^2)$?

5. Solve the following IVPs using the Laplace transform.

$$\begin{array}{lrcll} \text{(i)} & f'(x)+f(x) & = & x, & f(0)=0. \\ \text{(ii)} & f''(x)-f(x) & = & 4\mathrm{e}^x, & f(0)=f'(0)=1. \end{array}$$

6. (i) Show that the Laplace transform of x^a, where $a>-1$ is a real number, is $\Gamma(a+1)/p^{a+1}$ where the Gamma Function is defined as $\Gamma(s)=\int_0^\infty t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t$.

(ii) Find the Laplace transform of $(1-\cos(ax))/x$.

(iii) Find the Laplace transform of $\int_0^x \frac{\sin t}{t}\,\mathrm{d}t$.

7. (i) Solve the IVP in Exercise 2(ii) using the Laplace transform.

(ii) Use the Laplace transform to find a solution of

$$xf''(x)+2f'(x)+xf(x)=0.$$

Find a second independent solution of the equation. Why was this solution not found using the Laplace transform?

8.(Optional) A sequence of distributions (F_n) converges to a distribution F if $\langle F_n,\phi\rangle\to\langle F,\phi\rangle$ for all test functions ϕ.

(i) Show that if $F_n\to F$ then $F_n'\to F'$.

(ii) This limiting process applies to the partial sums (n terms) of a series. Define $F(x)=x$ for $-\pi<x<\pi$, extended periodically to $\mathbb{R}$. Show that its Fourier series is

$$F(x)=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kx.$$

Differentiate [the partial sums of] both sides to find an expression for $\sum_{k=1}^{\infty}(-1)^{k+1}\cos kx$. (Such a result has no counterpart in 'ordinary' analysis.)

(iii) Suppose that the integrable function $F(x)$ satisfies $\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x = 1$. (This implies that $\lim_{X\to\infty}\int_X^{\infty} F(x)\,\mathrm{d}x = 0$.) Define $F_n(x) = nF(nx)$. Draw a sketch to show how F_n is related to F. Show that $\langle F_n, \phi\rangle \to \phi(0)$ as $n \to \infty$. [Hint: split the range of integration into $(-\infty, -1/\sqrt{n})$, $(-1/\sqrt{n}, 1/\sqrt{n})$, $(1/\sqrt{n}, \infty)$; use the note above on the outer intervals and the MVT for integrals on the inner one.] Deduce that $F_n \to \delta$.

(iv) Suppose the random variable $X \sim N(0, \sigma^2)$ and write $G_\sigma(x)$ for its density function. Let $F_\sigma(x) = 2G_{2\sigma}(x) - G_\sigma(x)$. Show that F_σ satisfies the conditions of part (iii). What is $\lim_{\sigma\to 0^+} F_\sigma$? Roughly sketch $F_\sigma(x)$ for small σ and comment on your graph [hint: evaluate $F_\sigma(0)$]. Repeat for $F_\sigma(x) = 2G_{3\sigma}(x) - G_\sigma(x)$.

INTEGRAL TRANSFORMS – SHEET 2

(Exercises on lectures in Weeks 3 and 4)

Applications to ODEs. The Convolution and Inversion Theorem. Fourier Transform. Applications to PDEs.

1. The life time T of a particular brand of light bulb is modelled as follows. There is a probability p of the light-bulb blowing immediately (so that $T = 0$); given that the light bulb does not blow immediately, the probability of it having life time τ or less is $1 - \mathrm{e}^{-\lambda\tau}$ (where $\lambda > 0$).

(i) Write down the cumulative distribution function, $F_T(t)$, of T.

(ii) Write down the (generalized) probability density function $f_T(t)$ of T.

(iii) What is the expectation of T?

(iv) Write down the characteristic function of T, that is $\mathbb{E}(\mathrm{e}^{\mathrm{i}sT}) = \hat{f}_T(-s)$.

2. The *Laguerre polynomials* $L_n(x)$ are defined by

$$L_n(x) = \mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n \mathrm{e}^{-x}).$$

Show that $\overline{L_n}(p) = n!(p-1)^n p^{-n-1}$ and hence determine $L_n(x)$ for $1 \leqslant n \leqslant 4$.

3. Solve using Laplace transform methods the following differential and integral equations:

$$\text{(i)} \qquad f'(x) + f(x) = \mathbb{1}_{[0,1]}(x), \qquad f(0) = 0.$$

$$\text{(ii)} \qquad f'(x) - 2\int_0^x f(t)\mathrm{e}^{t-x}\mathrm{d}t = \mathrm{e}^{2x}, \qquad f(0) = 0.$$

4. Find the inverse Laplace transform of $\left(p^3+1\right)^{-1}$

(i) using partial fractions.

(ii) using the inversion formula.

(iii) using term-by-term inversion of power series.

[Hint for (iii): to find $\sum_{n=0}^{\infty} z^{3n}/(3n)!$, let $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$ so that $\omega^3 = 1$, note that $1 + \omega + \omega^2 = 0$, and consider $\mathrm{e}^z + \mathrm{e}^{\omega z} + \mathrm{e}^{\omega^2 z}$. You can adapt this technique to find the sum you need in (iii).]

5. In lectures it was shown that the Fourier transform of $f = \mathbb{1}_{[-1.1]}$ is $\hat{f}(s) = 2\sin s/s$. Determine the convolution $f * f$ and hence evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x.$$

6. The function $u(x,t)$ is defined for $x \in \mathbb{R}$ and $t > 0$ and solves the following boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = g(x).$$

Show that the Fourier transform $\hat{u}(s,t)$ of u in the x variable satisfies

$$\frac{\partial \hat{u}}{\partial t} = -ks^2\hat{u}, \qquad \hat{u}(s,0) = \hat{g}(s).$$

Deduce that

$$\hat{u}(s,t) = \hat{g}(s)\mathrm{e}^{-ks^2 t},$$

and hence write down the solution $u(x,t)$ as a convolution.

7. The function $u(x,y)$ is defined for $x \geqslant 0, y \geqslant 1$ and solves the following boundary value problem

$$y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = 1, \qquad u(x,1) = 1 = u(0,y).$$

Show that the Laplace transform $\overline{u}(p,y)$ of u in the x variable satisfies

$$y\frac{\partial \overline{u}}{\partial y} + p\overline{u} = \frac{1}{p} + 1, \qquad \overline{u}(p,1) = \frac{1}{p}.$$

Show further that $\overline{u}(p,y) = p^{-2} + p^{-1} - p^{-2}y^{-p}$ and deduce that

$$u(x,y) = \begin{cases} 1 + x & \text{if } \mathrm{e}^x < y \\ 1 + \log y & \text{if } \mathrm{e}^x \geqslant y. \end{cases}$$