Part A Fluid Dynamics and Waves: Sheet 3

1. Define the term *conformal map.*

   Write down conformal maps from the wedge $0 < \arg(z) < \alpha$, from the strip $-a < y < a$ and from the semi-infinite strip $0 < y < a, x > 0$ onto the upper half-plane. Find all the points at which each map is not conformal.

2. Show that the strength of a source or of a vortex is unaltered by a conformal transformation of the complex plane.

3. Fluid occupies the region between two plane rigid boundaries at $y = \pm b$, and there is a line vortex of strength Γ at the origin $(x, y) = (0, 0)$. Find the complex potential for the resulting flow (i) by the method of images; (ii) by using the conformal mapping $\zeta = \mathrm{e}^{\alpha z}$ with suitably chosen $\alpha > 0$.

4. Show that the circle $|\zeta| = a$ is mapped to a line segment

   $$S = \{z : \operatorname{Im} z = 0, -2a \leq \operatorname{Re} z \leq 2a\}$$

   along the real-z-axis by the Joukowski transformation

   $$z = \zeta + \frac{a^2}{\zeta}.$$

   Deduce that the exterior of the line segment S is mapped to the exterior of the circle $|\zeta| = a$ by the transformation

   $$\zeta = \frac{1}{2}\left(z + \sqrt{z^2 - 4a^2}\right). \qquad (\star)$$

   Carefully define the function $\sqrt{z^2 - 4a^2}$ and determine where the mapping $(\star)$ is conformal.

5. Show that the complex potential for a uniform stream of magnitude U aligned at an angle α to the real-ζ-axis with circulation Γ around a stationary circular cylinder of radius a centred on the origin is given by

   $$W(\zeta) = U\left(\zeta \mathrm{e}^{-\mathrm{i}\alpha} + \frac{a^2 \mathrm{e}^{\mathrm{i}\alpha}}{\zeta}\right) - \frac{\mathrm{i}\Gamma}{2\pi}\log\zeta.$$

   Hence find the complex potential $w(z)$ for flow past a flat plate at angle of incidence α in a uniform stream of magnitude U with circulation Γ. Deduce that the velocity at the trailing edge of the plate is finite only if the circulation satisfies the *Kutta condition*[1]

   $$\Gamma + 4\pi U a \sin\alpha = 0.$$

   Hence use the Kutta–Joukowski Lift Theorem to find the drag and lift forces experienced by the plate.

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[1] *hint: note that* $\dfrac{\mathrm{d}w}{\mathrm{d}z} \equiv \dfrac{\mathrm{d}W}{\mathrm{d}\zeta} \Big/ \dfrac{\mathrm{d}z}{\mathrm{d}\zeta}$

6. Two vortices, of strengths Γ_1 and Γ_2, are at the points $z = z_1$ and $z = z_2$ respectively in the complex plane. Write down the equations of motion for the position vectors $z_1(t)$ and $z_2(t)$ if the vortices are free to move. Assuming that $\Gamma_1 + \Gamma_2 \neq 0$, show that $\mathrm{d}Z/\mathrm{d}t = \mathrm{d}a/\mathrm{d}t = 0$, where

$$Z = \frac{\Gamma_1 z_1 + \Gamma_2 z_2}{\Gamma_1 + \Gamma_2}$$

is the *centroid* of the two vortices, and $a = |z_1 - z_2|$ is the distance between them.

Deduce that each vortex moves in a circle centred on Z, with angular velocity

$$\Omega = \frac{\Gamma_1 + \Gamma_2}{2\pi a^2}.$$

What happens in the exceptional case where $\Gamma_1 + \Gamma_2 = 0$?

7. Fluid occupies the region $x^2+y^2 > a^2$ outside a circular obstacle of radius a. By using the Circle Theorem, find the resulting complex potential when a vortex of strength Γ is placed at $(x, y) = (b, 0)$, where $b > a$ (assuming there to be no circulation about the obstacle).

Explain why the vortex will move in a circle of radius b with angular velocity of magnitude

$$\Omega = \frac{\Gamma a^2}{2\pi b^2 \left(b^2 - a^2\right)}.$$

8. Fluid occupies the quadrant $x > 0$, $y > 0$ bounded by two rigid boundaries along the x- and y-axes. Find the complex potential for the flow caused by a vortex at a point $z = c = a + ib$ in the fluid. If the vortex is free to move, show that it follows a path on which

$$\frac{1}{x^2} + \frac{1}{y^2} = \text{constant}.$$

9. [**Harder**] Fluid occupies the semi-infinite channel $\{z : \operatorname{Re} z > 0, -a < \operatorname{Im} z < a\}$. Show that the flow induced by a line vortex of strength $\Gamma > 0$ at the point $z = d \in \mathbb{R}^+$ has complex potential

$$w(z) = \frac{\mathrm{i}\Gamma}{2\pi}\left\{-\log\left[\sinh\left(\frac{\pi z}{2a}\right) - \sinh\left(\frac{\pi d}{2a}\right)\right] + \log\left[\sinh\left(\frac{\pi z}{2a}\right) + \sinh\left(\frac{\pi d}{2a}\right)\right]\right\}.$$

Show that the velocity components satisfy

$$u - \mathrm{i}v = \frac{\mathrm{i}\Gamma}{4a}\left\{\operatorname{cosech}\left(\frac{\pi(z+d)}{2a}\right) - \operatorname{cosech}\left(\frac{\pi(z-d)}{2a}\right)\right\}.$$

Deduce that, if the vortex is free to move, it will instantaneously travel downwards with speed $(\Gamma/4a)\operatorname{cosech}(\pi d/a)$.

[*Hint: use the identity* $\sinh^2(x) - \sinh^2(y) \equiv \sinh(x+y)\sinh(x-y)$.]