Part A Fluid Dynamics and Waves: Sheet 4

1. The free surface of a fluid moving in two dimensions is given parametrically by $\boldsymbol{r}(x,t) = \big(x, \eta(x,t)\big)$. Show that a unit normal to the surface is

$$\boldsymbol{n} = \frac{1}{\sqrt{1+\eta_x^2}} \left(-\eta_x, 1\right),$$

and deduce that the velocity of the surface normal to itself is given by

$$\frac{\partial \boldsymbol{r}}{\partial t} \cdot \boldsymbol{n} = \frac{\eta_t}{\sqrt{1+\eta_x^2}}.$$

Hence show that the kinematic condition that *the velocity of the fluid normal to the surface equals the velocity of the surface normal to itself* leads to the boundary condition

$$v = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \quad \text{on } y = \eta.$$

Deduce that *fluid particles on the free surface stay on the free surface.*

2. Consider small two-dimensional water waves on the free surface of an incompressible irrotational fluid with a velocity potential $\phi(x,y,t)$, which satisfies Laplaces equation. Suppose that the free surface has equation $y = \eta(x,t)$, the water has depth h, and the bottom is at $y = -h$. Show that we can choose ϕ such that the boundary conditions

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}, \qquad \frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\boldsymbol{\nabla}\phi\right|^2 + g\eta = 0$$

are satisfied on the free surface $y = \eta$. Show that, when the problem is linearised by neglecting quadratic terms, these boundary conditions are simplified to

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \qquad \frac{\partial \phi}{\partial t} + g\eta = 0$$

on $y = 0$. Show that travelling harmonic waves, with $\eta = A\cos(kx - \omega t)$ and $\phi = f(y)\sin(kx - \omega t)$, are possible provided $\omega^2 = gk\tanh(kh)$. Find and sketch the particle paths.

3. Inviscid incompressible fluid of density ρ_2 occupies the region $y > 0$ and lies vertically above a similar fluid of greater density ρ_1 in $y < 0$. Small amplitude waves perturb the interface between the fluids so that its equation becomes $y = \eta(x,t)$. Assuming η and the fluid velocities to be small, derive three boundary conditions relating η and the velocity potentials ϕ_1, ϕ_2 of the two fluids at $y = 0$. If $\eta(x,t) = A\cos(kx - \omega t)$, with $k > 0$, show that

$$\omega^2 = \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right) gk.$$

4. Suppose now that there is a surface tension T between the two fluids of question 3 and that $\rho_1 < \rho_2$. Derive the linearised boundary conditions to be satisfied at $y = 0$. Show that the frequency ω is now related to the wavenumber k by the equation

$$(\rho_1 + \rho_2)\,\omega^2 = k\left[Tk^2 - (\rho_2 - \rho_1)\,g\right].$$

Deduce that the waves are unstable if their wavelength λ exceeds a critical value

$$\lambda_{\mathrm{c}} = 2\pi\sqrt{\frac{T}{(\rho_2 - \rho_1)\,g}}\,.$$

5. Water flows steadily with speed U over a corrugated bed $y = -h + \varepsilon\cos(kx)$, where $\varepsilon \ll h$, so that there is a time-independent disturbance $\eta(x)$ to the free surface, which would be at $y = 0$ but for the corrugations. By writing the velocity components as

$$u = U + \frac{\partial\phi}{\partial x}, \qquad v = \frac{\partial\phi}{\partial y},$$

where $\phi(x, y)$ denotes the velocity potential of the disturbance to the uniform flow, show that the linearized boundary conditions are

$$\frac{\partial\phi}{\partial y} = U\frac{\mathrm{d}\eta}{\mathrm{d}x}, \qquad U\frac{\partial\phi}{\partial x} + g\eta = 0 \qquad \text{on } y = 0,$$

$$\frac{\partial\phi}{\partial y} = -Uk\varepsilon\sin(kx) \qquad \text{on } y = -h,$$

and hence find $\eta(x)$. Deduce that crests on the free surface occur immediately above troughs on the bed if

$$U^2 < \frac{g}{k}\tanh(kh),$$

but that crests on the surface overlie the crests on the bed if this inequality is reversed.