Numerical Analysis

Raphael Hauser
with thanks to Endre Süli

Oxford Mathematical Institute

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Lagrange interpolation

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More generally, we have the following result.

Theorem

Let $n \geq 0$. Then, $\exists p_n \in \Pi_n$ such that $p_n(x_i) = f_i$ for $i = 0, 1, \ldots, n$.

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$$L_{n,k}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} \in \Pi_n. \quad (1.1)$$

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Then

$$L_{n,k}(x_i) = 0 \text{ for } i = 0, \ldots, k-1, k+1, \ldots, n \text{ and } L_{n,k}(x_k) = 1.$$

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Hence

$$p_n(x_i) = \sum_{k=0}^{n} f_k L_{n,k}(x_i) = f_i \text{ for } i = 0, 1, \ldots, n. \qquad \square$$

The polynomial (1.2) is called the **Lagrange interpolating polynomial**.

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Then, their difference $d_n = p_n - q_n \in \Pi_n$ satisfies $d_n(x_k) = 0$ for $k = 0, 1, \ldots, n$. i.e., d_n is a polynomial of degree at most n but has at least $n + 1$ distinct roots.

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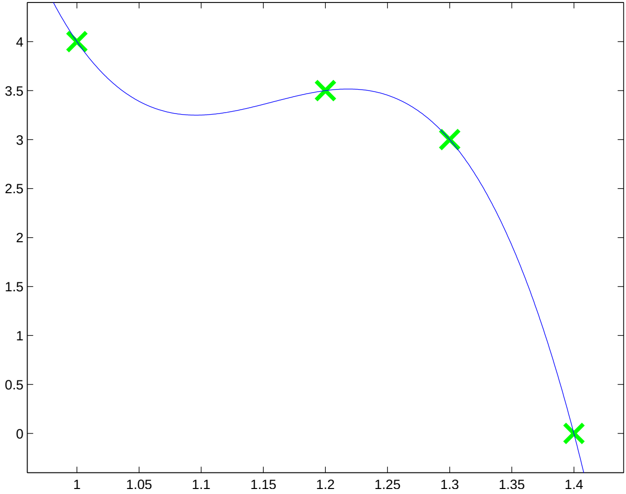
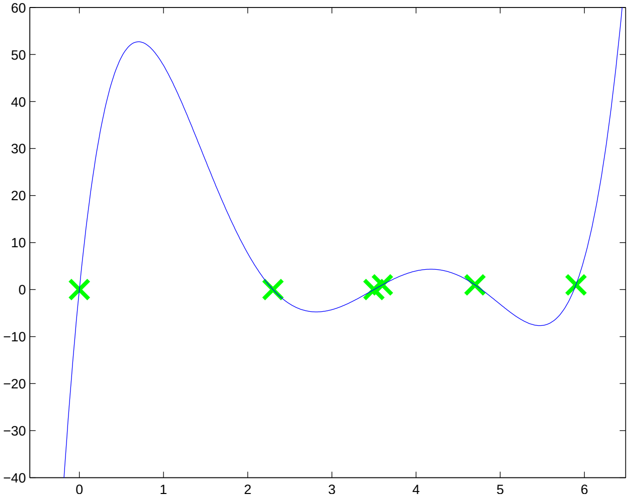
Algebra $\implies d_n \equiv 0 \implies p_n = q_n$. $\square$

Matlab:

```
% matlab
>> help lagrange
 LAGRANGE  Plots the Lagrange polynomial interpolant for the
 given DATA at the given KNOTS
>> lagrange([1,1.2,1.3,1.4],[4,3.5,3,0]);
```


4
3.5
3
2.5
2
1.5
1
0.5
0
1
1.05
1.1
1.15
1.2
1.25
1.3
1.35
1.4

```
>> lagrange([0,2.3,3.5,3.6,4.7,5.9],[0,0,0,1,1,1]);
```


60
50
40
30
20
10
0
−10
−20
−30
−40
0
1
2
3
4
5
6

Data from an underlying smooth function

Suppose that $f(x)$ has at least $n+1$ smooth derivatives in the interval (x_0, x_n). Let $f_k = f(x_k)$ for $k = 0, 1, \ldots, n$, and let p_n be the Lagrange interpolating polynomial for the data (x_k, f_k), $k = 0, 1, \ldots, n$.

Error: how large can the error $f(x) - p_n(x)$ be on the interval $[x_0, x_n]$?

Theorem

For every $x \in [x_0, x_n]$ *there exists* $\xi = \xi(x) \in (x_0, x_n)$ *such that*

$$e(x) \stackrel{\text{def}}{=} f(x) - p_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!},$$

where $f^{(n+1)}$ *is the* $n+1$*-st derivative of* f*.*

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Proof. Trivial for $x = x_k$, $k \in \{0, 1, \ldots, n\}$, as $e(x) = 0$ by construction. So suppose $x \neq x_k$. Let

$$\phi(t) \stackrel{\text{def}}{=} e(t) - \frac{e(x)}{\pi(x)}\pi(t),$$

where

$$\begin{aligned} \pi(t) &\stackrel{\text{def}}{=} (t - x_0)(t - x_1)\cdots(t - x_n) \\ &= t^{n+1} - \left(\sum_{i=0}^{n} x_i\right) t^n + \cdots (-1)^{n+1} x_0 x_1 \cdots x_n \\ &\in \Pi_{n+1}. \end{aligned}$$

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But

$$\phi^{(n+1)}(t) = e^{(n+1)}(t) - \frac{e(x)}{\pi(x)}\pi^{(n+1)}(t) = f^{(n+1)}(t) - \frac{e(x)}{\pi(x)}(n+1)!$$

since $p_n^{(n+1)}(t) \equiv 0$ and because $\pi(t)$ is a monic polynomial of degree $n+1$.

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The result follows immediately from this identity since $\phi^{(n+1)}(\xi) = 0$. $\square$

Example: $f(x) = \log(1+x)$ on $[0,1]$. Here,

$$|f^{(n+1)}(\xi)| = n!/(1+\xi)^{n+1} < n!$$

on $(0,1)$. So

$$|e(x)| < |\pi(x)|n!/(n+1)! \leq 1/(n+1)$$

since $|x - x_k| \leq 1$ for each x, x_k, $k = 0, 1, \ldots, n$, in $[0,1] \implies |\pi(x)| \leq 1$.

This is probably pessimistic for many x, e.g. for $x = \frac{1}{2}$, $\pi(\frac{1}{2}) \leq 2^{-(n+1)}$ as $|\frac{1}{2} - x_k| \leq \frac{1}{2}$.

This example shows the important fact that the error can be large at the end points of the interval.
There is a famous example due to Runge, where the error from the interpolating polynomial approximation to

$$f(x) = (1 + x^2)^{-1}$$

for $n + 1$ equally-spaced points on $[-5, 5]$ diverges near ± 5 as $n \to \infty$.

Matlab: try `runge` from the website.