Numerical Analysis

Raphael Hauser
with thanks to Endre Süli

Oxford Mathematical Institute

HT 2019

Building Lagrange interpolating polynomials from lower degree ones

Building Lagrange interpolating polynomials from lower degree ones

Theorem

Let $Q_{i,j}$ be the Lagrange interpolating polynomial at x_k, $k = i, \ldots, j$. Then,

$$Q_{i,j}(x) = \frac{(x - x_i)Q_{i+1,j}(x) - (x - x_j)Q_{i,j-1}(x)}{x_j - x_i} \tag{0.1}$$

Building Lagrange interpolating polynomials from lower degree ones

Theorem

Let $Q_{i,j}$ be the Lagrange interpolating polynomial at x_k, $k = i, \ldots, j$. Then,

$$Q_{i,j}(x) = \frac{(x - x_i)Q_{i+1,j}(x) - (x - x_j)Q_{i,j-1}(x)}{x_j - x_i} \tag{0.1}$$

Proof. Let $s(x)$ denote the right-hand side of (0.1). Because of uniqueness, we simply wish to show that $s(x_k) = f_k$.

Building Lagrange interpolating polynomials from lower degree ones

Theorem

Let $Q_{i,j}$ be the Lagrange interpolating polynomial at x_k, $k = i, \ldots, j$. Then,

$$Q_{i,j}(x) = \frac{(x - x_i)Q_{i+1,j}(x) - (x - x_j)Q_{i,j-1}(x)}{x_j - x_i} \tag{0.1}$$

Proof. Let $s(x)$ denote the right-hand side of (0.1). Because of uniqueness, we simply wish to show that $s(x_k) = f_k$. For $k = i + 1, \ldots, j - 1$, $Q_{i+1,j}(x_k) = f_k = Q_{i,j-1}(x_k)$, and hence

$$s(x_k) = \frac{(x_k - x_i)Q_{i+1,j}(x_k) - (x_k - x_j)Q_{i,j-1}(x_k)}{x_j - x_i} = f_k.$$

Building Lagrange interpolating polynomials from lower degree ones

Theorem

Let $Q_{i,j}$ be the Lagrange interpolating polynomial at x_k, $k = i, \ldots, j$. Then,

$$Q_{i,j}(x) = \frac{(x - x_i)Q_{i+1,j}(x) - (x - x_j)Q_{i,j-1}(x)}{x_j - x_i} \tag{0.1}$$

Proof. Let $s(x)$ denote the right-hand side of (0.1). Because of uniqueness, we simply wish to show that $s(x_k) = f_k$. For $k = i+1, \ldots, j-1$, $Q_{i+1,j}(x_k) = f_k = Q_{i,j-1}(x_k)$, and hence

$$s(x_k) = \frac{(x_k - x_i)Q_{i+1,j}(x_k) - (x_k - x_j)Q_{i,j-1}(x_k)}{x_j - x_i} = f_k.$$

We also have that $Q_{i+1,j}(x_j) = f_j$ and $Q_{i,j-1}(x_i) = f_i$, and hence

$$s(x_i) = Q_{i,j-1}(x_i) = f_i \text{ and } s(x_j) = Q_{i+1,j}(x_j) = f_j.$$ □

Comment

This result can be used as the basis for constructing interpolating polynomials. In books: may find topics such as the Newton form and divided differences.

Generalisation

Generalisation

Given data f_i, g_i at distinct x_i, $i = 0, 1, \ldots, n$, with $x_0 < x_1 < \cdots < x_n$, can we find a polynomial p such that $p(x_i) = f_i$ and $p'(x_i) = g_i$?

Generalisation

Given data f_i, g_i at distinct x_i, $i = 0, 1, \ldots, n$, with $x_0 < x_1 < \cdots < x_n$, can we find a polynomial p such that $p(x_i) = f_i$ and $p'(x_i) = g_i$?

Theorem

There is a unique polynomial $p \in \Pi_{2n+1}$ such that $p(x_i) = f_i$ and $p'(x_i) = g_i$ for $i = 0, 1, \ldots, n$.

Generalisation

Given data f_i, g_i at distinct x_i, $i = 0, 1, \ldots, n$, with $x_0 < x_1 < \cdots < x_n$, can we find a polynomial p such that $p(x_i) = f_i$ and $p'(x_i) = g_i$?

Theorem

There is a unique polynomial $p \in \Pi_{2n+1}$ such that $p(x_i) = f_i$ and $p'(x_i) = g_i$ for $i = 0, 1, \ldots, n$.

Construction: given $L_{n,k}(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$, let

$$\begin{aligned} H_{n,k}(x) &= [L_{n,k}(x)]^2(1 - 2(x - x_k)L'_{n,k}(x_k)) \\ \text{and} \quad K_{n,k}(x) &= [L_{n,k}(x)]^2(x - x_k). \end{aligned}$$

Then **Hermite interpolating polynomial**

$$p_{2n+1}(x) = \sum_{k=0}^{n}[f_k H_{n,k}(x) + g_k K_{n,k}(x)] \tag{0.2}$$

interpolates the data as required.

Theorem

Let h_{2n+1} *be the Hermite interpolating polynomial in the case where* $f_i = f(x_i)$ *and* $g_i = f'(x_i)$ *and* f *has at least* $2n+2$ *smooth derivatives. Then, for every* $x \in [x_0, x_n]$,

$$f(x) - h_{2n+1}(x) = [(x - x_0)(x - x_{k-1}) \cdots (x - x_n)]^2 \frac{f^{(2n+2)}(\xi)}{(2n+2)!},$$

where $\xi \in (x_0, x_n)$ *and* $f^{(2n+2)}$ *is the* $2n+2$*-nd derivative of* f.

Theorem

Let h_{2n+1} be the Hermite interpolating polynomial in the case where $f_i = f(x_i)$ and $g_i = f'(x_i)$ and f has at least $2n+2$ smooth derivatives. Then, for every $x \in [x_0, x_n]$,

$$f(x) - h_{2n+1}(x) = [(x - x_0)(x - x_{k-1}) \cdots (x - x_n)]^2 \frac{f^{(2n+2)}(\xi)}{(2n+2)!},$$

where $\xi \in (x_0, x_n)$ and $f^{(2n+2)}$ is the $2n+2$-nd derivative of f.

Compare with the error formula we obtained for Lagrange interpolation:

Theorem

For every $x \in [x_0, x_n]$ there exists $\xi = \xi(x) \in (x_0, x_n)$ such that

$$f(x) - p_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!},$$

where $f^{(n+1)}$ is the $n+1$-st derivative of f.

Newton–Cotes Quadrature

Terminology: Quadrature $\equiv$ numerical integration

Setup: given $f(x_k)$ at $n+1$ equally spaced points $x_k = x_0 + k \cdot h$, $k = 0, 1, \ldots, n$, where $h = (x_n - x_0)/n$. Suppose that $p_n \in \Pi_n$ interpolates this data.

Idea: does

$$\int_{x_0}^{x_n} f(x)\,\mathrm{d}x \approx \int_{x_0}^{x_n} p_n(x)\,\mathrm{d}x? \tag{2.3}$$

We investigate the error in such an approximation below, but note that

$$\begin{aligned}\int_{x_0}^{x_n} p_n(x)\,\mathrm{d}x &= \int_{x_0}^{x_n} \sum_{k=0}^{n} f(x_k)\cdot L_{n,k}(x)\,\mathrm{d}x \\ &= \sum_{k=0}^{n} f(x_k)\cdot \int_{x_0}^{x_n} L_{n,k}(x)\,\mathrm{d}x \\ &= \sum_{k=0}^{n} w_k f(x_k),\end{aligned} \tag{2.4}$$

where the coefficients

$$w_k = \int_{x_0}^{x_n} L_{n,k}(x)\,\mathrm{d}x \tag{2.5}$$

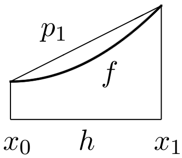
$k = 0, 1, \ldots, n$, are independent of f.

A formula

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{k=0}^{n} w_k f(x_k)$$

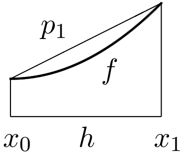
with $x_k \in [a, b]$ and w_k independent of f for $k = 0, 1, \ldots, n$ is called a **quadrature formula**; the coefficients w_k are known as **weights**; the points x_k are called the **quadrature points**. The specific form (2.3)–(2.5) is called a **Newton–Cotes formula** of order n.

Trapezium Rule: $n = 1$:



$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}x \approx \frac{h}{2}[f(x_0) + f(x_1)]$$

Trapezium Rule: $n = 1$:



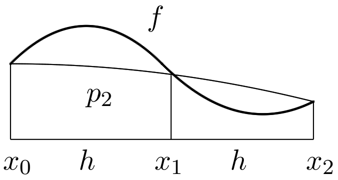
$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}x \approx \frac{h}{2}[f(x_0) + f(x_1)]$$

Proof.

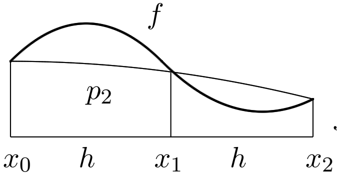
$$\int_{x_0}^{x_1} p_1(x)\,\mathrm{d}x = f(x_0)\int_{x_0}^{x_1} \overbrace{\frac{x - x_1}{x_0 - x_1}}^{L_{1,0}(x)}\,\mathrm{d}x + f(x_1)\int_{x_0}^{x_1} \overbrace{\frac{x - x_0}{x_1 - x_0}}^{L_{1,1}(x)}\,\mathrm{d}x$$

$$= f(x_0)\frac{(x_1 - x_0)}{2} + f(x_1)\frac{(x_1 - x_0)}{2}$$

Simpson's Rule: $n = 2$:

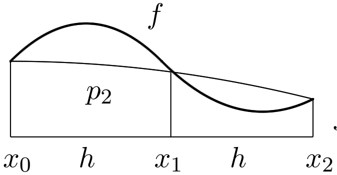


Simpson's Rule: $n = 2$:



$$\begin{aligned}\int_{x_0}^{x_2} f(x)\,\mathrm{d}x &\approx \tfrac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \\ &= \int_{x_0}^{x_2} p_2(x)\,\mathrm{d}x \\ &= \sum_{k=0}^{2} f(x_k) \cdot \int_{x_0}^{x_2} L_{2,k}(x)\,\mathrm{d}x.\end{aligned}$$

Simpson's Rule: $n = 2$:



$$\int_{x_0}^{x_2} f(x)\,\mathrm{d}x \approx \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$
$$= \int_{x_0}^{x_2} p_2(x)\,\mathrm{d}x$$
$$= \sum_{k=0}^{2} f(x_k) \cdot \int_{x_0}^{x_2} L_{2,k}(x)\,\mathrm{d}x.$$

Note: The Trapezium Rule is exact if $f \in \Pi_1$, since if $f \in \Pi_1 \implies p_1 = f$. Similarly, Simpson's Rule is exact if $f \in \Pi_2$, since if $f \in \Pi_2 \implies p_2 = f$. The highest degree of polynomial exactly integrated by a quadrature rule is called the **degree of accuracy**.

Error: we can use the error in interpolation directly to obtain

$$\int_{x_0}^{x_n} [f(x) - p_n(x)] \, \mathrm{d}x = \int_{x_0}^{x_n} \frac{\pi(x)}{(n+1)!} f^{(n+1)}(\xi(x)) \, \mathrm{d}x$$

Error: we can use the error in interpolation directly to obtain

$$\int_{x_0}^{x_n} [f(x) - p_n(x)] \, \mathrm{d}x = \int_{x_0}^{x_n} \frac{\pi(x)}{(n+1)!} f^{(n+1)}(\xi(x)) \, \mathrm{d}x$$

so that

$$\left| \int_{x_0}^{x_n} [f(x) - p_n(x)] \, \mathrm{d}x \right| \leq \frac{1}{(n+1)!} \max_{\xi \in [x_0, x_n]} |f^{(n+1)}(\xi)| \int_{x_0}^{x_n} |\pi(x)| \, \mathrm{d}x, \tag{2.6}$$

Error: we can use the error in interpolation directly to obtain

$$\int_{x_0}^{x_n} [f(x) - p_n(x)]\,\mathrm{d}x = \int_{x_0}^{x_n} \frac{\pi(x)}{(n+1)!} f^{(n+1)}(\xi(x))\,\mathrm{d}x$$

so that

$$\left|\int_{x_0}^{x_n} [f(x) - p_n(x)]\,\mathrm{d}x\right| \leq \frac{1}{(n+1)!} \max_{\xi\in[x_0,x_n]} |f^{(n+1)}(\xi)| \int_{x_0}^{x_n} |\pi(x)|\,\mathrm{d}x, \tag{2.6}$$

which, e.g., for the Trapezium Rule, $n = 1$, gives

$$\left|\int_{x_0}^{x_1} f(x)\,\mathrm{d}x - \frac{(x_1 - x_0)}{2}[f(x_0) + f(x_1)]\right| \leq \frac{(x_1 - x_0)^3}{12} \max_{\xi\in[x_0,x_1]} |f''(\xi)|.$$

Error: we can use the error in interpolation directly to obtain

$$\int_{x_0}^{x_n} [f(x) - p_n(x)]\,\mathrm{d}x = \int_{x_0}^{x_n} \frac{\pi(x)}{(n+1)!} f^{(n+1)}(\xi(x))\,\mathrm{d}x$$

so that

$$\left|\int_{x_0}^{x_n} [f(x) - p_n(x)]\,\mathrm{d}x\right| \le \frac{1}{(n+1)!} \max_{\xi\in[x_0,x_n]} |f^{(n+1)}(\xi)| \int_{x_0}^{x_n} |\pi(x)|\,\mathrm{d}x, \tag{2.6}$$

which, e.g., for the Trapezium Rule, $n = 1$, gives

$$\left|\int_{x_0}^{x_1} f(x)\,\mathrm{d}x - \frac{(x_1 - x_0)}{2}[f(x_0) + f(x_1)]\right| \le \frac{(x_1 - x_0)^3}{12} \max_{\xi\in[x_0,x_1]} |f''(\xi)|.$$

In fact, we can prove a tighter result using the Integral Mean-Value Theorem.

Theorem (Integral Mean-Value Theorem)

Suppose that f and g are continuous on $[a, b]$ and $g(x) \geq 0$ on this interval. Then, there exits an $\eta \in (a, b)$ for which

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(\eta)\int_a^b g(x)\,\mathrm{d}x.$$

Theorem (Integral Mean-Value Theorem)

Suppose that f and g are continuous on $[a,b]$ and $g(x) \geq 0$ on this interval. Then, there exits an $\eta \in (a,b)$ for which

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(\eta)\int_a^b g(x)\,\mathrm{d}x.$$

Theorem

Suppose f'' is continuous on (x_0, x_1). Then, there exists $\xi \in (x_0, x_1)$ such that

$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}x - \frac{(x_1 - x_0)}{2}[f(x_0) + f(x_1)] = -\frac{(x_1 - x_0)^3}{12} f''(\xi).$$

Proof. See problem sheet. □

For $n > 1$, (2.6) gives pessimistic bounds. For example, in the case $n = 2$, corresponding to Simpson's Rule, the bound becomes

$$\left| \int_{x_0}^{x_2} [f(x) - p_2(x)] \, \mathrm{d}x \right| \leq \frac{(x_2 - x_0)^4}{192} \max_{\xi \in [x_0, x_2]} |f'''(\xi)|.$$

For $n > 1$, (2.6) gives pessimistic bounds. For example, in the case $n = 2$, corresponding to Simpson's Rule, the bound becomes

$$\left| \int_{x_0}^{x_2} [f(x) - p_2(x)] \, \mathrm{d}x \right| \le \frac{(x_2 - x_0)^4}{192} \max_{\xi \in [x_0, x_2]} |f'''(\xi)|.$$

We can prove a better result:

Theorem (Error in Simpson's Rule)

Suppose that f'''' is continuous on $[x_0, x_2]$. Then,

$$\left| \int_{x_0}^{x_2} f(x) \, \mathrm{d}x - \frac{(x_2 - x_0)}{6} [f(x_0) + 4f(x_1) + f(x_2)] \right| \le \frac{(x_2 - x_0)^5}{720} \max_{\xi \in [x_0, x_2]} |f''''(\xi)|.$$

The proof relies on the following result.

Theorem (Intermediate-Value Theorem)

Suppose that f is continuous on a closed interval $[a, b]$, and c is any number between $f(a)$ and $f(b)$ inclusive. Then, there is at least one number ξ in the closed interval such that $f(\xi) = c$.

The proof relies on the following result.

Theorem (Intermediate-Value Theorem)

Suppose that f is continuous on a closed interval $[a, b]$, and c is any number between $f(a)$ and $f(b)$ inclusive. Then, there is at least one number ξ in the closed interval such that $f(\xi) = c$.

In particular, since $c = (df(a) + ef(b))/(d + e)$ lies between $f(a)$ and $f(b)$ for any positive d and e, there is a value ξ in the closed interval for which $d \cdot f(a) + e \cdot f(b) = (d + e) \cdot f(\xi)$.

Proof. Recall

$$\int_{x_0}^{x_2} p_2(x)\,\mathrm{d}x = \tfrac{1}{3}h[f(x_0) + 4f(x_1) + f(x_2)],$$

where $h = x_2 - x_1 = x_1 - x_0$.

Proof. Recall

$$\int_{x_0}^{x_2} p_2(x)\,\mathrm{d}x = \tfrac{1}{3}h[f(x_0) + 4f(x_1) + f(x_2)],$$

where $h = x_2 - x_1 = x_1 - x_0$.

Consider $f(x_0) - 2f(x_1) + f(x_2) = f(x_1 - h) - 2f(x_1) + f(x_1 + h)$.

Proof. Recall

$$\int_{x_0}^{x_2} p_2(x)\,\mathrm{d}x = \tfrac{1}{3}h[f(x_0) + 4f(x_1) + f(x_2)],$$

where $h = x_2 - x_1 = x_1 - x_0$.

Consider $f(x_0) - 2f(x_1) + f(x_2) = f(x_1 - h) - 2f(x_1) + f(x_1 + h)$.

Then, by Taylor's Theorem,

$$\begin{array}{c} f(x_1 - h) \\ -2f(x_1) \\ +f(x_1 + h) \end{array} = \begin{array}{l} f(x_1) - hf'(x_1) + \frac{1}{2}h^2 f''(x_1) - \frac{1}{6}h^3 f'''(x_1) + \frac{1}{24}h^4 f''''(\xi_1) \\ -2f(x_1) + \\ f(x_1) + hf'(x_1) + \frac{1}{2}h^2 f''(x_1) + \frac{1}{6}h^3 f'''(x_1) + \frac{1}{24}h^4 f''''(\xi_2) \end{array}$$

for some $\xi_1 \in (x_0, x_1)$ and $\xi_2 \in (x_1, x_2)$,

Proof. Recall

$$\int_{x_0}^{x_2} p_2(x)\,\mathrm{d}x = \tfrac{1}{3}h[f(x_0) + 4f(x_1) + f(x_2)],$$

where $h = x_2 - x_1 = x_1 - x_0$.
Consider $f(x_0) - 2f(x_1) + f(x_2) = f(x_1 - h) - 2f(x_1) + f(x_1 + h)$.
Then, by Taylor's Theorem,

$$\begin{array}{ccl} f(x_1 - h) & & f(x_1) - hf'(x_1) + \frac{1}{2}h^2 f''(x_1) - \frac{1}{6}h^3 f'''(x_1) + \frac{1}{24}h^4 f''''(\xi_1) \\ -2f(x_1) & = & -2f(x_1) + \\ +f(x_1 + h) & & f(x_1) + hf'(x_1) + \frac{1}{2}h^2 f''(x_1) + \frac{1}{6}h^3 f'''(x_1) + \frac{1}{24}h^4 f''''(\xi_2) \end{array}$$

for some $\xi_1 \in (x_0, x_1)$ and $\xi_2 \in (x_1, x_2)$, and hence

$$\begin{aligned} f(x_0) - 2f(x_1) + f(x_2) &= h^2 f''(x_1) + \tfrac{1}{24}h^4[f''''(\xi_1) + f''''(\xi_2)] \\ &= h^2 f''(x_1) + \tfrac{1}{12}h^4 f''''(\xi_3), \end{aligned} \tag{2.7}$$

the last result following from the Intermediate-Value Theorem for some $\xi_3 \in (\xi_1, \xi_2) \subset (x_0, x_2)$.

Now for any $x \in [x_0, x_2]$, we may use Taylor's Theorem again to deduce

Now for any $x \in [x_0, x_2]$, we may use Taylor's Theorem again to deduce

$$\begin{aligned}\int_{x_0}^{x_2} f(x)\,\mathrm{d}x = {} & f(x_1)\int_{x_1-h}^{x_1+h}\mathrm{d}x + f'(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)\,\mathrm{d}x \\ & + \tfrac{1}{2}f''(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)^2\,\mathrm{d}x + \tfrac{1}{6}f'''(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)^3\,\mathrm{d}x \\ & + \tfrac{1}{24}\int_{x_1-h}^{x_1+h} f''''(\eta_1(x))(x-x_1)^4\,\mathrm{d}x \quad \text{where } \eta_1(x) \in (x_0, x_2),\end{aligned}$$

Now for any $x \in [x_0, x_2]$, we may use Taylor's Theorem again to deduce

$$\begin{aligned}\int_{x_0}^{x_2} f(x)\,\mathrm{d}x = &\; f(x_1)\int_{x_1-h}^{x_1+h} \mathrm{d}x + f'(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)\,\mathrm{d}x \\ &+ \tfrac{1}{2}f''(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)^2\,\mathrm{d}x + \tfrac{1}{6}f'''(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)^3\,\mathrm{d}x \\ &+ \tfrac{1}{24}\int_{x_1-h}^{x_1+h} f''''(\eta_1(x))(x-x_1)^4\,\mathrm{d}x \quad \text{where } \eta_1(x) \in (x_0, x_2), \\ \overset{\text{IMVT}}{=} &\quad 2hf(x_1) + \tfrac{1}{3}h^3 f''(x_1) + \tfrac{1}{60}h^5 f''''(\eta_2), \\ &\quad \text{where } \eta_2 \in (x_0, x_2) \text{ by the Integral Mean Value Theorem,}\end{aligned}$$

Now for any $x \in [x_0, x_2]$, we may use Taylor's Theorem again to deduce

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)\,\mathrm{d}x = {} & f(x_1)\int_{x_1-h}^{x_1+h}\mathrm{d}x + f'(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)\,\mathrm{d}x \\
& + \tfrac{1}{2}f''(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)^2\,\mathrm{d}x + \tfrac{1}{6}f'''(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)^3\,\mathrm{d}x \\
& + \tfrac{1}{24}\int_{x_1-h}^{x_1+h} f''''(\eta_1(x))(x-x_1)^4\,\mathrm{d}x \quad \text{where } \eta_1(x) \in (x_0, x_2), \\
\overset{\text{IMVT}}{=} {} & 2hf(x_1) + \tfrac{1}{3}h^3 f''(x_1) + \tfrac{1}{60}h^5 f''''(\eta_2), \\
& \text{where } \eta_2 \in (x_0, x_2) \text{ by the Integral Mean Value Theorem,} \\
\overset{(2.7)}{=} {} & \tfrac{1}{3}h[f(x_0) + 4f(x_1) + f(x_2)] + \tfrac{1}{60}h^5 f''''(\eta_2) - \tfrac{1}{36}h^5 f''''(\xi_3)
\end{aligned}$$

Now for any $x \in [x_0, x_2]$, we may use Taylor's Theorem again to deduce

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)\,\mathrm{d}x = {} & f(x_1)\int_{x_1-h}^{x_1+h} \mathrm{d}x + f'(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)\,\mathrm{d}x \\
& + \tfrac{1}{2}f''(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)^2\,\mathrm{d}x + \tfrac{1}{6}f'''(x_1)\int_{x_1-h}^{x_1+h}(x-x_1)^3\,\mathrm{d}x \\
& + \tfrac{1}{24}\int_{x_1-h}^{x_1+h} f''''(\eta_1(x))(x-x_1)^4\,\mathrm{d}x \quad \text{where } \eta_1(x) \in (x_0, x_2), \\
\overset{\text{IMVT}}{=} {} & 2hf(x_1) + \tfrac{1}{3}h^3 f''(x_1) + \tfrac{1}{60}h^5 f''''(\eta_2), \\
& \text{where } \eta_2 \in (x_0, x_2) \text{ by the Integral Mean Value Theorem,} \\
\overset{(2.7)}{=} {} & \tfrac{1}{3}h[f(x_0) + 4f(x_1) + f(x_2)] + \tfrac{1}{60}h^5 f''''(\eta_2) - \tfrac{1}{36}h^5 f''''(\xi_3)
\end{aligned}$$

for some $\xi_3 \in (\xi_1, \xi_2) \subset (x_0, x_2)$, having replaced $h^2 f''(x_1)$ from (2.7).

In summary, for any $x \in [x_0, x_2]$,

$$\int_{x_0}^{x_2} f(x)\,\mathrm{d}x = \tfrac{1}{3}h[f(x_0) + 4f(x_1) + f(x_2)] + \tfrac{1}{60}h^5 f''''(\eta_2) - \tfrac{1}{36}h^5 f''''(\xi_3)$$
$$= \int_{x_0}^{x_2} p_2(x)\,\mathrm{d}x + \frac{1}{180}\left(\frac{x_2 - x_0}{2}\right)^5 \left(3f''''(\eta_2) - 5f''''(\xi_3)\right).$$

In summary, for any $x \in [x_0, x_2]$,

$$\int_{x_0}^{x_2} f(x)\,\mathrm{d}x = \tfrac{1}{3}h[f(x_0) + 4f(x_1) + f(x_2)] + \tfrac{1}{60}h^5 f''''(\eta_2) - \tfrac{1}{36}h^5 f''''(\xi_3)$$
$$= \int_{x_0}^{x_2} p_2(x)\,\mathrm{d}x + \frac{1}{180}\left(\frac{x_2 - x_0}{2}\right)^5 \left(3f''''(\eta_2) - 5f''''(\xi_3)\right).$$

Thus, taking moduli, and using the Intermediate-Value Theorem,

$$\left|\int_{x_0}^{x_2} [f(x) - p_2(x)]\,\mathrm{d}x\right| \le \frac{8}{2^5 \cdot 180}(x_2 - x_0)^5 \max_{\xi \in [x_0, x_2]} |f''''(\xi)|. \qquad \square$$

Note: Simpson's Rule is exact if $f \in \Pi_3$ since then $f'''' \equiv 0$.

In fact, it is possible to compute a slightly stronger bound.

Theorem (Error in Simpson's Rule II)

Suppose that f'''' is continuous on (x_0, x_2). Then,

$$\int_{x_0}^{x_2} f(x)\,\mathrm{d}x = \frac{x_2 - x_0}{6}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{(x_2 - x_0)^5}{2880} f''''(\xi)$$

for some $\xi \in (x_0, x_2)$.

Proof. See Süli and Mayers, Thm. 7.2.