Numerical Analysis

Raphael Hauser
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Oxford Mathematical Institute

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LU Factorization

The basic operation of Gaussian Elimination, row $i \leftarrow$ row $i + \lambda *$ row j can be achieved by pre-multiplication by a special lower-triangular matrix

$$M(i,j,\lambda) = I + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow i$$
$$\begin{matrix} \uparrow \\ j \end{matrix}$$

where I is the identity matrix.

Example: $n = 4$,

$$M(3,2,\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } M(3,2,\lambda) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ \lambda b + c \\ d \end{pmatrix},$$

i.e., $M(3,2,\lambda)A$ performs: row 3 of $A \leftarrow$ row 3 of $A + \lambda *$ row 2 of A and similarly $M(i,j,\lambda)A$ performs: row i of $A \leftarrow$ row i of $A + \lambda *$ row j of A.

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So GE for e.g., $n = 3$ is

$$\underset{l_{32} = \frac{a_{32}}{a_{22}}}{M(3,2,-l_{32})} \cdot \underset{l_{31} = \frac{a_{31}}{a_{11}}}{M(3,1,-l_{31})} \cdot \underset{l_{21} = \frac{a_{21}}{a_{11}}}{M(2,1,-l_{21})} \cdot A = U = (\,\diagdown\,).$$

upper triangular

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The l_{ij} are the **multipliers**.

Be careful:

each multiplier l_{ij} uses the data a_{ij} and a_{ii} that *results from the transformations already applied*, not data from the original matrix.

So l_{32} uses a_{32} and a_{22} that result from the previous transformations $M(2, 1, -l_{21})$ and $M(3, 1, -l_{31})$.

Example.

$$M(2,1,-l_{21})A = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 2 \\ 0 & \frac{7}{3} & 2 \\ 2 & -2 & -1 \end{pmatrix},$$

$$M(3,1,-l_{21})\,[M(2,1,-l_{21})A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 \\ 0 & \frac{7}{3} & 2 \\ 2 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 2 \\ 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & -\frac{4}{3} & -\frac{7}{3} \end{pmatrix}$$

$$M(3,2,-l_{32})\,[M(3,1,-l_{21})M(2,1,-l_{21})A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{4}{7} & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 \\ 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & -\frac{4}{3} & -\frac{7}{3} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 2 \\ 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & 0 & -1 \end{pmatrix}$$

$$M(3,2,-l_{32}M(3,1,-l_{21})M(2,1,-l_{21}) = \begin{pmatrix} 1 & 0 & 0 \\ -0.3333 & 1 & 0 \\ -0.8571 & 0.5714 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -0.3333 & 1 & 0 \\ -0.8571 & 0.5714 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0.3333 & 1 & 0 \\ 0.6667 & -0.5714 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & -1 & 2 \\ 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & 0 & -1 \end{pmatrix}$$

$$A = LU$$

Lemma

If $i \neq j$, $(M(i,j,\lambda))^{-1} = M(i,j,-\lambda)$.

Proof. Exercise.

Outcome: for $n = 3$, $A = M(2,1,l_{21}) \cdot M(3,1,l_{31}) \cdot M(3,2,l_{32}) \cdot U$, where

$$M(2,1,l_{21}) \cdot M(3,1,l_{31}) \cdot M(3,2,l_{32}) = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} = L = (◺).$$

lower

triangular

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Theorem

For any dimension n, GE can be expressed as $A = LU$, where $U = (\,\ulcorner\!\!\diagdown\,)$ is upper triangular resulting from GE, and $L = (\,\llcorner\!\!\diagdown\,)$ is unit lower triangular (lower triangular with ones on the diagonal) with l_{ij} = multiplier used to create the zero in the (i,j)th position.

Most implementations of GE therefore, rather than doing GE as above,

$$\begin{aligned}
\text{factorize} \quad & A = LU && \text{(takes } \approx \tfrac{1}{3}n^3 \text{ flops)} \\
\text{and then solve} \quad & Ax = b && \\
\text{by solving} \quad & Ly = b && \text{(forward substitution)} \\
\text{and then} \quad & Ux = y && \text{(back substitution)}
\end{aligned}$$

Note: this is much more efficient if we have many different right-hand sides b but the same A.

Pivoting: GE or LU can fail if the pivot $a_{ii} = 0$, e.g., if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

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$$\begin{aligned} 0 \cdot x_1 + 1 \cdot x_2 &= 1 \\ 1 \cdot x_1 + 0 \cdot x_2 &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} 1 \cdot x_1 + 0 \cdot x_2 &= 2 \\ 0 \cdot x_1 + 1 \cdot x_2 &= 1 \end{aligned}$$

are the same, but their matrices

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$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

have had their rows reordered: GE fails for the first but succeeds for the second $\Longrightarrow$ better to interchange the rows and then apply GE.

Partial pivoting:

when creating the zeros in the j-th column, find

$$|a_{kj}| = \max(|a_{jj}|, |a_{j+1j}|, \dots, |a_{nj}|),$$

then swap (interchange) rows j and k

e.g.,

$$\begin{pmatrix} 4 & 2 & 5 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -2 & -1 \\ 0 & 3 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 4 & 2 & 5 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -2 & -1 \end{pmatrix}$$

$$M(3,2,-l_{3,2}) \begin{pmatrix} 4 & 2 & 5 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 5 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 & 5 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & 2 & -2 & -1 \end{pmatrix}$$

$$M(4,2,-l_{4,2}) \begin{pmatrix} 4 & 2 & 5 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & 2 & -2 & -1 \end{pmatrix} = \ldots$$

more generally,

$$\begin{pmatrix} a_{11} & \cdot & a_{1j-1} & a_{1j} & \cdot & \cdot & \cdot & a_{1n} \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & a_{j-1j-1} & a_{j-1j} & \cdot & \cdot & \cdot & a_{j-1n} \\ 0 & \cdot & 0 & a_{jj} & \cdot & \cdot & \cdot & a_{jn} \\ 0 & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & a_{kj} & \cdot & \cdot & \cdot & a_{kn} \\ 0 & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & a_{nj} & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix}$$

$\rightarrow$

$$\begin{pmatrix} a_{11} & \cdot & a_{1j-1} & a_{1j} & \cdot & \cdot & \cdot & a_{1n} \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & a_{j-1j-1} & a_{j-1j} & \cdot & \cdot & \cdot & a_{j-1n} \\ 0 & \cdot & 0 & a_{kj} & \cdot & \cdot & \cdot & a_{kn} \\ 0 & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & a_{jj} & \cdot & \cdot & \cdot & a_{jn} \\ 0 & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & a_{nj} & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix}$$

Theorem

GE with partial pivoting cannot fail if A is non singular.

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Proof. If A is the first matrix above at the j-th stage,

$$\det[A] = a_{11} \cdots a_{j-1j-1} \cdot \det \begin{pmatrix} a_{jj} & \cdot & \cdot & \cdot & a_{jn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{kj} & \cdot & \cdot & \cdot & a_{kn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{nj} & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix}.$$

Hence $\det[A] = 0$ if $a_{jj} = \cdots = a_{kj} = \cdots = a_{nj} = 0$. Thus if the pivot $a_{k,j}$ is zero, A is singular. So if all of the pivots are nonzero, A is nonsingular. (Note, actually a_{nn} can be zero and an LU factorization still exist.)

The effect of pivoting is just a permutation (reordering) of the rows, and hence can be represented by a permutation matrix P.

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$$PA = LU$$

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For example,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} A$$

just has the 2nd row of A first, the 3rd row of A second and the 1st row of A last.

Matlab:

```
% matlab
>> A = rand(5,5)
A =
    0.8147    0.0975    0.1576    0.1419    0.6557
    0.9058    0.2785    0.9706    0.4218    0.0357
    0.1270    0.5469    0.9572    0.9157    0.8491
    0.9134    0.9575    0.4854    0.7922    0.9340
    0.6324    0.9649    0.8003    0.9595    0.6787
>> b = A*ones(5,1)
b =
    1.8675
    2.6124
    3.3959
    4.0825
    4.0358
```

```
>> x = A \ b

x =
    1.0000
    1.0000
    1.0000
    1.0000
    1.0000


>> [L,U,P] = lu(A)
```

```
L =

   1.0000         0         0         0         0
   0.8920    1.0000         0         0         0
   0.1390   -0.5469    1.0000         0         0
   0.9917    0.8870    0.9924    1.0000         0
   0.6923   -0.3991    0.4794    0.1476    1.0000


U =

   0.9134    0.9575    0.4854    0.7922    0.9340
        0   -0.7565   -0.2753   -0.5648   -0.1774
        0         0    0.7391    0.4967    0.6223
        0         0         0   -0.3559   -1.3507
        0         0         0         0   -0.1376

P =

     0     0     0     1     0
     1     0     0     0     0
     0     0     1     0     0
     0     1     0     0     0
     0     0     0     0     1
```

```
>> P*P'
ans =
     1     0     0     0     0
     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
     0     0     0     0     1

 >> norm(L*U-P*A)


ans =


   1.6214e-16
```