Numerical Analysis

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Best Approximation in Inner-Product Spaces

Best approximation of functions:

given a function f defined on $[a, b]$, find the "closest"

- polynomial, or
- piecewise polynomial (see later sections), or
- trigonometric polynomial (truncated Fourier series).

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What do we mean by "closest"?

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Given a vector space V over the field $\mathbb{R}$ of real numbers, the mapping $\|\cdot\| : V \to \mathbb{R}$ is a **norm** on V if it satisfies the following axioms:

1. $\|f\| \geq 0$ for all $f \in V$, with $\|f\| = 0$ if, and only if, $f = 0 \in V$;
2. $\|\lambda f\| = |\lambda|\|f\|$ for all $\lambda \in \mathbb{R}$ and all $f \in V$; and
3. $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in V$ (the **triangle inequality**).

Examples of norms on $\mathbb{R}^n$

1. For vectors $x \in \mathbb{R}^n$, with $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$,

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

   is the ℓ^1- or vector one-norm.

2. For vectors $x \in \mathbb{R}^n$, with $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$,

$$\|x\|_2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} = \sqrt{x^{\mathrm{T}} x}$$

   is the ℓ^2- or vector two-norm.

3. For vectors $x \in \mathbb{R}^n$, with $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$,

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

   is the ℓ^∞- or vector infinity-norm.

Examples of norms on function spaces

1. For integrable functions on (a, b),
$$\|f\|_1 = \int_a^b |f(x)| \, \mathrm{d}x$$
is the L^1- or one-norm.
2. For functions in
$$V = \mathrm{L}^2(a, b) \equiv \{f : (a, b) \to \mathbb{R} \mid \int_a^b [f(x)]^2 \, \mathrm{d}x < \infty\}$$
we define
$$\|f\|_2 = \left(\int_a^b [f(x)]^2 \, \mathrm{d}x \right)^{1/2},$$
the L^2- or two-norm.
3. For continuous functions on $[a, b]$,
$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$$
is the L^∞- or ∞-norm.

Weighted L^2 norm

Suppose that w is a real-valued function, defined, positive and integrable on (a, b). Consider the vector space

$$V = \mathrm{L}^2_w(a, b) \equiv \{f : (a, b) \to \mathbb{R} \mid \int_a^b w(x)[f(x)]^2 \, \mathrm{d}x < \infty\}$$

(this certainly includes continuous functions on $[a, b]$, and piecewise continuous functions on $[a, b]$ with finitely many jump-discontinuities), equipped with the (weighted) L^2- or (weighted) two-norm

$$\|f\| \equiv \|f\|_2 = \left(\int_a^b w(x)[f(x)]^2 \, \mathrm{d}x \right)^{1/2} .$$

Special case: If $w(x) \equiv 1$, then $\mathrm{L}^2_w(a, b) = \mathrm{L}^2(a, b)$.

Least-squares polynomial approximation: aim to find the best polynomial approximation to $f \in \mathrm{L}^2_w(a,b)$, i.e., find $p_n \in \Pi_n$ for which

$$\|f - p_n\|_2 \leq \|f - q\|_2 \qquad \forall q \in \Pi_n.$$

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Seeking p_n in the form $p_n(x) = \sum_{k=0}^{n} \alpha_k x^k$ then results in the minimization problem

$$\min_{(\alpha_0,\ldots,\alpha_n)} \int_a^b w(x) \left[f(x) - \sum_{k=0}^{n} \alpha_k x^k \right]^2 \mathrm{d}x.$$

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The unique minimizer can be found from the (linear) system

$$\frac{\partial}{\partial \alpha_j} \int_a^b w(x) \left[f(x) - \sum_{k=0}^{n} \alpha_k x^k \right]^2 \mathrm{d}x = 0 \ \text{ for each } \ j = 0, 1, \ldots, n.$$

[Exercise: Why? How?] But there is important additional structure here.

Inner-product spaces: a real **inner-product space** is a vector space V over $\mathbb{R}$ with a mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ (the **inner product**) for which

1. $\langle v, v \rangle \geq 0$ for all $v \in V$ and $\langle v, v \rangle = 0$ if, and only if, $v = 0$;
2. $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$; and
3. $\langle \alpha u + \beta v, z \rangle = \alpha \langle u, z \rangle + \beta \langle v, z \rangle$ for all $u, v, z \in V$ and all $\alpha, \beta \in \mathbb{R}$.

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   where $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, \ldots, y_n)^{\mathrm{T}}$.

2. $V = \mathrm{L}^2_w(a, b) = \{f : (a, b) \to \mathbb{R} \mid \int_a^b w(x)[f(x)]^2 \, \mathrm{d}x < \infty\}$,

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x) \, \mathrm{d}x,$$

   where $f, g \in \mathrm{L}^2_w(a, b)$, and where w is a weight-function, defined, positive and integrable on (a, b).

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1. Suppose that V is an inner-product space with inner product $\langle \cdot, \cdot \rangle$. Then, $v \in V \mapsto \langle v, v \rangle^{1/2} \in \mathbb{R}$ defines a norm on V (see below for a proof): in Example 2 above, it is the (weighted) L^2-norm.
2. Suppose that V is an inner-product space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ defined by this inner product via $\|v\| := \langle v, v \rangle^{1/2}$. The angle θ between $u, v \in V$ is

$$\theta = \cos^{-1}\left(\frac{\langle u, v \rangle}{\|u\|\|v\|}\right).$$

Thus, u and v are orthogonal in $V \iff \langle u, v \rangle = 0$.

Example:

x^2 and $\frac{3}{4} - x$ are orthogonal in $\mathrm{L}^2(0,1)$ with $\langle f, g \rangle := \int_0^1 f(x)g(x)\,\mathrm{d}x$ as

$$\int_0^1 x^2(\tfrac{3}{4} - x)\,\mathrm{d}x = \tfrac{1}{4} - \tfrac{1}{4} = 0.$$

Exercise:

Find $f \in \Pi_2$ such that f is orthogonal to each $g \in \Pi_1$ in the inner product

$$\langle f, g \rangle := \int_{-1}^1 x^4 f(x)\,g(x)\,\mathrm{d}x$$

and $f(1) = \frac{2}{7}$.

Pythagoras Theorem

Suppose that V is an inner-product space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ defined by this inner product. For any $u, v \in V$ such that $\langle u, v \rangle = 0$ we have

$$\|u \pm v\|^2 = \|u\|^2 + \|v\|^2.$$

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Proof.

$$\begin{aligned}
\|u \pm v\|^2 &= \langle u \pm v, u \pm v \rangle = \langle u, u \pm v \rangle \pm \langle v, u \pm v \rangle && \text{[axiom (iii)]} \\
&= \langle u, u \pm v \rangle \pm \langle u \pm v, v \rangle && \text{[axiom (ii)]} \\
&= \langle u, u \rangle \pm \langle u, v \rangle \pm \langle u, v \rangle + \langle v, v \rangle \\
&= \langle u, u \rangle + \langle v, v \rangle && \text{[orthogonality]} \\
&= \|u\|^2 + \|v\|^2.
\end{aligned}$$

□

Cauchy–Schwarz inequality

Suppose that V is an inner-product space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ defined by this inner product. For any $u, v \in V$,

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Proof. For every $\lambda \in \mathbb{R}$,

$$0 \leq \langle u - \lambda v, u - \lambda v \rangle = \|u\|^2 - 2\lambda \langle u, v \rangle + \lambda^2 \|v\|^2 = \phi(\lambda),$$

which is a quadratic in λ. The minimizer of ϕ is at $\lambda_* = \langle u, v \rangle / \|v\|^2$, and thus since $\phi(\lambda_*) \geq 0$, $\|u\|^2 - \langle u, v \rangle^2 / \|v\|^2 \geq 0$, which gives the required inequality. □

Triangle inequality

Suppose that V is an inner-product space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ defined by this inner product. For any $u, v \in V$,

$$\|u + v\| \leq \|u\| + \|v\|. \tag{10.1}$$

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Proof. Note that

$$\|u + v\|^2 = \langle u + v, u + v \rangle = \|u\|^2 + 2\langle u, v \rangle + \|v\|^2.$$

Hence, by the Cauchy–Schwarz inequality,

$$\|u + v\|^2 \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2 .$$

Taking square-roots yields (10.1). □

Note: The function $\|\cdot\| : V \to \mathbb{R}$ defined by $\|v\| := \langle v, v \rangle^{1/2}$ on the inner-product space V, with inner product $\langle \cdot, \cdot \rangle$, trivially satisfies the first two axioms of norm on V; this is a consequence of $\langle \cdot, \cdot \rangle$ being an inner product on V. Result (10.1) above implies that $\|\cdot\|$ also satisfies the third axiom of norm, the triangle inequality.