Numerical Analysis

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as a basis for Π_n, suppose we have a basis

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We shall choose the basis $\{\phi_0, \phi_1, \ldots, \phi_n\}$ so as to ensure that the matrix A in the normal equations is diagonal.

With this choice of basis, we seek the polynomial of best approximation $p_n(x) = \sum_{k=0}^{n} \beta_k \phi_k(x)$. The normal equations then become

$$\int_a^b w(x) \left(f(x) - \sum_{k=0}^{n} \beta_k \phi_k(x) \right) \phi_i(x) \, \mathrm{d}x = 0, \quad i = 0, 1, \ldots, n,$$

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or equivalently

$$\sum_{k=0}^{n} \left(\int_a^b w(x) \phi_k(x) \phi_i(x) \, \mathrm{d}x \right) \beta_k = \int_a^b w(x) f(x) \phi_i(x) \mathrm{d}x, \quad i = 0, 1, \ldots, n,$$

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i.e.,

$$A\beta = \varphi, \tag{12.1}$$

where $\beta = (\beta_0, \beta_1, \ldots, \beta_n)^{\mathrm{T}}$, $\varphi = (f_1, f_2, \ldots, f_n)^{\mathrm{T}}$ and now

$$a_{i,k} = \int_a^b w(x) \phi_k(x) \phi_i(x)\, \mathrm{d}x \qquad \text{and} \qquad f_i = \int_a^b w(x) f(x) \phi_i(x)\, \mathrm{d}x.$$

So A is diagonal if

$$\langle \phi_i, \phi_k \rangle = \int_a^b w(x)\phi_i(x)\phi_k(x)\,\mathrm{d}x \quad \begin{cases} = 0 & i \neq k \text{ and} \\ \neq 0 & i = k. \end{cases}$$

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We can create such a set of **orthogonal polynomials**

$$\{\phi_0, \phi_1, \ldots, \phi_n, \ldots\},$$

with $\phi_i \in \Pi_i$ for each i, by the Gram–Schmidt procedure, which is based on the following lemma.

Lemma

Suppose that $\{\phi_0, \phi_1, \ldots, \phi_k\}$*, with* $\phi_i \in \Pi_i$ *for each* i*, are orthogonal with respect to the inner product* $\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, \mathrm{d}x$*. Then,*

$$\phi_{k+1}(x) = x^{k+1} - \sum_{i=0}^{k} \lambda_i \phi_i(x)$$

satisfies

$$\langle \phi_{k+1}, \phi_j \rangle = \int_a^b w(x) \phi_{k+1}(x) \phi_j(x)\, \mathrm{d}x = 0, \qquad j = 0, 1, \ldots, k,$$

when

$$\lambda_j = \frac{\langle x^{k+1}, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}, \qquad j = 0, 1, \ldots, k.$$

Proof. For any j, $0 \leq j \leq k$,

$$\begin{aligned} \langle \phi_{k+1}, \phi_j \rangle = & \langle x^{k+1}, \phi_j \rangle - \sum_{i=0}^{k} \lambda_i \langle \phi_i, \phi_j \rangle \\ = & \langle x^{k+1}, \phi_j \rangle - \lambda_j \langle \phi_j, \phi_j \rangle \end{aligned}$$

by the orthogonality of ϕ_i and ϕ_j, $i \neq j$,

$$= 0$$ by definition of λ_j. □

Notes:

1. The Gram–Schmidt procedure does the above for $k = 0, 1, \ldots, n$ successively.
2. ϕ_k is always of exact degree k, so $\{\phi_0, \phi_1, \ldots, \phi_\ell\}$ is a basis for Π_ℓ for every $\ell \geq 0$.
3. ϕ_k can be normalised/scaled to satisfy $\langle \phi_k, \phi_k \rangle = 1$ or to be monic.

Examples:

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1. The inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$$

has orthogonal polynomials called the **Legendre polynomials**,

$$\phi_0(x) \equiv 1, \;\; \phi_1(x) = x, \;\; \phi_2(x) = x^2 - \tfrac{1}{3}, \;\; \phi_3(x) = x^3 - \tfrac{3}{5}x, \ldots$$

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2. The inner product

$$\langle f, g \rangle = \int_{-1}^{1} \frac{f(x) g(x)}{\sqrt{1 - x^2}} \, \mathrm{d}x$$

gives orthogonal polynomials, which are the **Chebyshev polynomials**,

$$\phi_0(x) \equiv 1, \;\; \phi_1(x) = x, \;\; \phi_2(x) = 2x^2 - 1, \;\; \phi_3(x) = 4x^3 - 3x, \ldots$$

3. The inner product

$$\langle f, g\rangle = \int_0^\infty \mathrm{e}^{-x} f(x) g(x)\,\mathrm{d}x$$

gives orthogonal polynomials, which are the **Laguerre polynomials**,

$$\phi_0(x) \equiv 1,\ \ \phi_1(x) = 1 - x,\ \ \phi_2(x) = 2 - 4x + x^2,$$

$$\phi_3(x) = 6 - 18x + 9x^2 - x^3, \ldots$$

Lemma

Suppose that $\{\phi_0, \phi_1, \ldots, \phi_n, \ldots\}$ *are orthogonal polynomials for a given inner product* $\langle \cdot, \cdot \rangle$*. Then,* $\langle \phi_k, q \rangle = 0$ *whenever* $q \in \Pi_{k-1}$*.*

Lemma

Suppose that $\{\phi_0, \phi_1, \ldots, \phi_n, \ldots\}$ *are orthogonal polynomials for a given inner product* $\langle \cdot, \cdot \rangle$. *Then,* $\langle \phi_k, q \rangle = 0$ *whenever* $q \in \Pi_{k-1}$.

Proof. This follows since if $q \in \Pi_{k-1}$, then $q(x) = \sum_{i=0}^{k-1} \sigma_i \phi_i(x)$ for some $\sigma_i \in \mathbb{R}$, $i = 0, 1, \ldots, k-1$, so

$$\langle \phi_k, q \rangle = \sum_{i=0}^{k-1} \sigma_i \langle \phi_k, \phi_i \rangle = 0.$$

□

Remark: note from the above argument that if $q(x) = \sum_{i=0}^{k} \sigma_i \phi_i(x)$ is of exact degree k (so $\sigma_k \neq 0$), then $\langle \phi_k, q \rangle = \sigma_k \langle \phi_k, \phi_k \rangle \neq 0$.

Theorem

Suppose that $\{\phi_0, \phi_1, \ldots, \phi_n, \ldots\}$ *is a set of orthogonal polynomials. Then, there exist sequences of real numbers* $(\alpha_k)_{k=1}^{\infty}$, $(\beta_k)_{k=1}^{\infty}$, $(\gamma_k)_{k=1}^{\infty}$ *such that a three-term recurrence relation of the form*

$$\phi_{k+1}(x) = \alpha_k(x - \beta_k)\phi_k(x) - \gamma_k\phi_{k-1}(x), \qquad k = 1, 2, \ldots,$$

holds.

Proof. The polynomial $x\phi_k \in \Pi_{k+1}$, so there exist $\sigma_{k,0}, \sigma_{k,1}, \ldots, \sigma_{k,k+1}$ in $\mathbb{R}$ such that

$$x\phi_k(x) = \sum_{i=0}^{k+1} \sigma_{k,i}\phi_i(x)$$

as $\{\phi_0, \phi_1, \ldots, \phi_{k+1}\}$ is a basis for Π_{k+1}.

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as $\{\phi_0, \phi_1, \ldots, \phi_{k+1}\}$ is a basis for Π_{k+1}. Now take the inner product on both sides with ϕ_j, and note that $x\phi_j \in \Pi_{k-1}$ if $j \leq k-2$.

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as $\{\phi_0, \phi_1, \dots, \phi_{k+1}\}$ is a basis for Π_{k+1}. Now take the inner product on both sides with ϕ_j, and note that $x\phi_j \in \Pi_{k-1}$ if $j \leq k-2$. Note that

$$\langle x\phi_k, \phi_j \rangle = \int_a^b w(x)x\phi_k(x)\phi_j(x)\,\mathrm{d}x = \int_a^b w(x)\phi_k(x)x\phi_j(x)\,\mathrm{d}x = \langle \phi_k, x\phi_j \rangle = 0$$

by the above lemma for $j \leq k-2$.

Proof. The polynomial $x\phi_k \in \Pi_{k+1}$, so there exist $\sigma_{k,0}, \sigma_{k,1}, \ldots, \sigma_{k,k+1}$ in $\mathbb{R}$ such that

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as $\{\phi_0, \phi_1, \ldots, \phi_{k+1}\}$ is a basis for Π_{k+1}. Now take the inner product on both sides with ϕ_j, and note that $x\phi_j \in \Pi_{k-1}$ if $j \le k-2$. Note that

$$\langle x\phi_k, \phi_j\rangle = \int_a^b w(x)x\phi_k(x)\phi_j(x)\,\mathrm{d}x = \int_a^b w(x)\phi_k(x)x\phi_j(x)\,\mathrm{d}x = \langle \phi_k, x\phi_j\rangle = 0$$

by the above lemma for $j \le k-2$. In addition,

$$\left\langle \sum_{i=0}^{k+1} \sigma_{k,i}\phi_i, \phi_j \right\rangle = \sum_{i=0}^{k+1} \sigma_{k,i}\langle \phi_i, \phi_j\rangle = \sigma_{k,j}\langle \phi_j, \phi_j\rangle$$

by the linearity of $\langle \cdot, \cdot \rangle$ and orthogonality of ϕ_k and ϕ_j for $k \ne j$.

Hence $\sigma_{k,j} = 0$ for $j \leq k - 2$, and so

$$x\phi_k(x) = \sigma_{k,k+1}\phi_{k+1}(x) + \sigma_{k,k}\phi_k(x) + \sigma_{k,k-1}\phi_{k-1}(x).$$

Hence $\sigma_{k,j} = 0$ for $j \leq k-2$, and so

$$x\phi_k(x) = \sigma_{k,k+1}\phi_{k+1}(x) + \sigma_{k,k}\phi_k(x) + \sigma_{k,k-1}\phi_{k-1}(x).$$

Taking the inner product with ϕ_{k+1} reveals that

$$\langle x\phi_k, \phi_{k+1}\rangle = \sigma_{k,k+1}\langle \phi_{k+1}, \phi_{k+1}\rangle,$$

so $\sigma_{k,k+1} \neq 0$ by the above remark as $x\phi_k$ is of exact degree $k+1$.

Hence $\sigma_{k,j} = 0$ for $j \leq k-2$, and so

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so $\sigma_{k,k+1} \neq 0$ by the above remark as $x\phi_k$ is of exact degree $k+1$. Thus,

$$\phi_{k+1}(x) = \frac{1}{\sigma_{k,k+1}}(x - \sigma_{k,k})\phi_k(x) - \frac{\sigma_{k,k-1}}{\sigma_{k,k+1}}\phi_{k-1}(x),$$

which is of the given form, with

$$\alpha_k = \frac{1}{\sigma_{k,k+1}}, \qquad \beta_k = \sigma_{k,k}, \qquad \gamma_k = \frac{\sigma_{k,k-1}}{\sigma_{k,k+1}}, \qquad k = 1, 2, \ldots. \qquad \square$$

Example. The inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} f(x) g(x) \, \mathrm{d}x$$

has orthogonal polynomials called the **Hermite polynomials**,

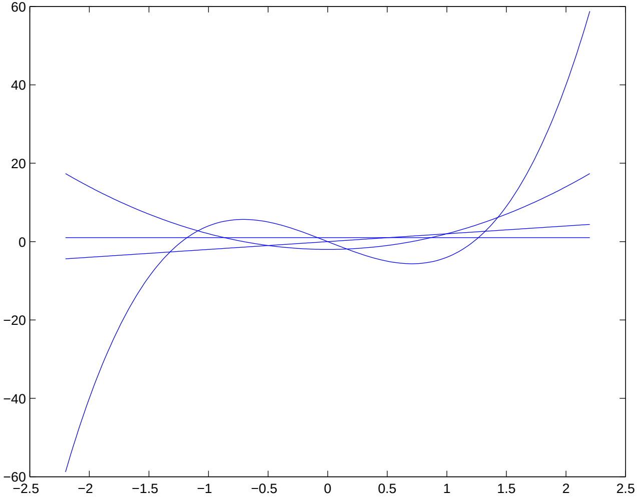
$$\phi_0(x) \equiv 1, \quad \phi_1(x) = 2x, \quad \phi_{k+1}(x) = 2x\phi_k(x) - 2k\phi_{k-1}(x) \text{ for } k \geq 1.$$

Matlab:

```
% cat hermite_polys.m

x=linspace(-2.2,2.2,200);
oldH=ones(1,200); plot(x,oldH), hold on
newH=2*x; plot(x,newH)
for n=1:2,...
  newnewH=2*x.*newH-2*n*oldH; plot(x,newnewH),...
  oldH=newH;newH=newnewH;
end

% matlab
>> hermite_polys
```


60
40
20
0
−20
−40
−60
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5