# Numerical Analysis

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with thanks to Endre Süli

Oxford Mathematical Institute

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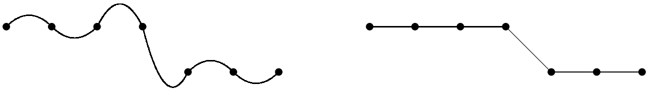
# Recap on Splines



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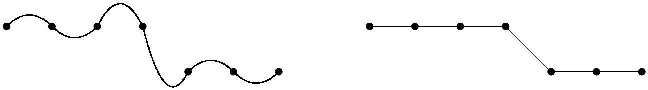

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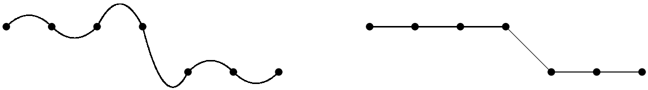

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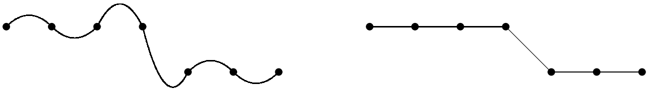

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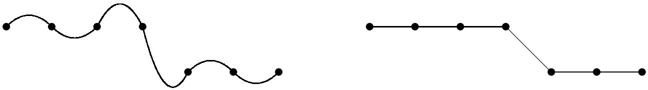

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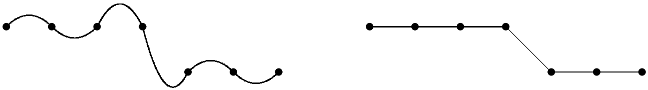

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- $C^1$ function (quadratic splines, not discussed further in this course),
- $C^2$ function (cubic splines, see below),
- $C^3$ ... (higher order splines, not discussed in this course).

## Theorem

*Let* $s(x)$ *be the linear spline interpolation of a function* $f \in \mathrm{C}^2[a,b]$ *at nodes* $x_0 < x_1 < \cdots < x_n$. *Then,*

$$\|f - s\|_\infty \leq \frac{1}{8} h^2 \|f''\|_\infty$$

*where* $h = \max\limits_{1 \leq i \leq n} (x_i - x_{i-1})$ *and* $\|f''\|_\infty = \max\limits_{x \in [a,b]} |f''(x)|$.

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$$s(x) = \begin{cases} \sum_{i=1}^{n} s_i(x) & \text{for } x \in [a,b] \setminus \{x_0, \ldots, x_n\}, \\ f(x_i) & \text{for } x = x_i, \quad (i = 0, \ldots, n). \end{cases}$$

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$$s_i(x_{i-1}^+) := \lim_{x \searrow x_{i-1}} s_i(x) = f(x_i), \quad (i = 1, \ldots, n)$$

$$s_i(x_i^-) := \lim_{x \nearrow x_i} s_i(x) = f(x_{i+1}), \quad (i = 1, \ldots, n),$$

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(B) first derivatives of adjacent patches must match at nodes, giving first order smoothness,

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(C) second derivatives of adjacent patches must match at nodes, giving second order smoothness,

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(c) enforce continuity of $s'''$ at $x_1$ and $x_{n-1}$ (which implies that the first two pieces are the same cubic spline, i.e., on $[x_0, x_2]$, and similarly for the last two pieces, i.e., on $[x_{n-2}, x_n]$, from which it follows that $x_1$ and $x_{n-1}$ are not knots! — this is usually described as the '**not a knot**' end-conditions).

We may write Conditions (A), (B) and (C) complemented with one of (a), (b) or (c) in matrix form as

$$Ay = g, \tag{14.1}$$

with

$$y = (a_1, b_1, c_1, d_1, a_2, \ldots, d_{n-1}, a_n, b_n, c_n, d_n)^{\mathrm{T}}$$

and the various entries of $g$ are $f(x_i)$, $i \in 0, 1, \ldots, n$, and $f'(x_0)$, $f'(x_n)$ for (a), and zeros for (b).

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So if $A$ is nonsingular, this implies that $y = A^{-1}g$, that is there is a unique set of coefficients $\{a_1, b_1, c_1, d_1, a_2, \ldots, d_{n-1}, a_n, b_n, c_n, d_n\}$.

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We now prove that if $Ay = 0$ then $y = 0$, and thus that $A$ is nonsingular for cases (a) and (b) — it is also possible, but more complicated, to show this for case (c).

### Theorem

*If* $f(x_i) = 0$ *at the knots* $x_i$, $i = 1, \ldots, n$, *and* $f'(x_0) = 0 = f'(x_n)$ *for case (a), then* $s(x) = 0$ *for all* $x \in [x_0, x_n]$.

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Proof. Consider

$$\begin{aligned}\int_{x_0}^{x_n} (s''(x))^2 \, \mathrm{d}x = & \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (s_i''(x))^2 \, \mathrm{d}x \\ = & \sum_{i=1}^{n} \left[ s_i'(x) s_i''(x) \right]_{x_{i-1}}^{x_i} - \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} s_i'(x) s_i'''(x) \, \mathrm{d}x\end{aligned}$$

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using integration by parts. However,

$$\int_{x_{i-1}}^{x_i} s_i'(x) s_i'''(x) \,\mathrm{d}x = s_i'''(x) \int_{x_{i-1}}^{x_i} s_i'(x) \,\mathrm{d}x = s_i'''(x) \left[s_i(x)\right]_{x_{i-1}}^{x_i} = 0$$

since $s_i'''(x)$ is constant on the interval $(x_{i-1}, x_i)$ and $s_i(x_{i-1}) = 0 = s_i(x_i)$.

Thus, matching first and second derivatives at the knots, telescopic cancellation gives

$$\begin{aligned}
\int_{x_0}^{x_n} (s''(x))^2 \,\mathrm{d}x &= \sum_{i=1}^{n} \left[ s_i'(x) s_i''(x) \right]_{x_{i-1}}^{x_i} \\
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&\quad + s_2'(x_2)s_2''(x_2) - s_2'(x_1)s_2''(x_1) + \cdots \\
&\quad + s_{n-1}'(x_{n-1})s_{n-1}''(x_{n-1}) - s_{n-1}'(x_{n-2})s_{n-1}''(x_{n-2}) \\
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### Proposition

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Proof. If $p, q \in \mathrm{C}^2[a, b]$ then $\alpha p + \beta q \in \mathrm{C}^2[a, b]$; also $p, q \in \Pi_3$ implies that $\alpha p + \beta q \in \Pi_3$ for every $\alpha$, $\beta \in \mathbb{R}$.

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Finally, the natural end-conditions (b) imply that

$$(\alpha p + \beta q)''(x_0) = 0 = (\alpha p + \beta q)''(x_n)$$

whenever $p''$ and $q''$ are zero at $x_0$ and $x_n$. $\square$

Best spline bases: the **Cardinal splines**, $C_i$, $i = 0, 1, \ldots, n$, defined as the interpolatory natural cubic splines satisfying

$$C_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases}$$

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Preferred are the **B-splines** (locally) defined by

$$B_i(x_i) = 1 \qquad \text{for } i = 2, 3, \dots, n-2,$$

$B_i(x) \equiv 0$ for $x \notin (x_{i-2}, x_{i+2})$, $B_i$ a cubic spline with knots $x_j$, $j = 0, 1, \dots, n$, with special definitions for $B_0$, $B_1$, $B_{n-1}$ and $B_n$.

Example/construction: Cubic $B$-spline with knots $0, 1, 2, 3, 4$. On $[0, 1]$,

$$B(x) = ax^3$$

for some $a$ in order that $B$, $B'$ and $B''$ are continuous at $x = 0$ (recall that $B(x)$ is required to be identically zero for $x < 0$). So

$$B(1) = a, \quad B'(1) = 3a, \quad \text{and} \quad B''(1) = 6a.$$

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On $[1, 2]$, since $B$ is a cubic polynomial, using Taylor's Theorem,

$$\begin{aligned} B(x) = \quad & B(1) + B'(1)(x-1) + \frac{B''(1)}{2}(x-1)^2 + \beta(x-1)^3 \\ = \quad & a + 3a(x-1) + 3a(x-1)^2 + \beta(x-1)^3 \end{aligned}$$

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for some $\beta$, and since we require $B(2) = 1$, then $\beta = 1 - 7a$. Now, in order to continue, by symmetry, we must have $B'(2) = 0$, i.e.,

$$3a + 6a(x-1)_{x=2} + 3(1-7a)(x-1)^2_{x=2} = 3 - 12a = 0$$

and hence $a = \frac{1}{4}$.

So

$$B(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{4}x^3 & \text{for } x \in [0,1] \\ -\frac{3}{4}(x-1)^3 + \frac{3}{4}(x-1)^2 + \frac{3}{4}(x-1) + \frac{1}{4} & \text{for } x \in [1,2] \\ -\frac{3}{4}(3-x)^3 + \frac{3}{4}(3-x)^2 + \frac{3}{4}(3-x) + \frac{1}{4} & \text{for } x \in [2,3] \\ \frac{1}{4}(4-x)^3 & \text{for } x \in [3,4] \\ 0 & \text{for } x > 4. \end{cases}$$

So

$$B(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{4}x^3 & \text{for } x \in [0,1] \\ -\frac{3}{4}(x-1)^3 + \frac{3}{4}(x-1)^2 + \frac{3}{4}(x-1) + \frac{1}{4} & \text{for } x \in [1,2] \\ -\frac{3}{4}(3-x)^3 + \frac{3}{4}(3-x)^2 + \frac{3}{4}(3-x) + \frac{1}{4} & \text{for } x \in [2,3] \\ \frac{1}{4}(4-x)^3 & \text{for } x \in [3,4] \\ 0 & \text{for } x > 4. \end{cases}$$

More generally: $B$-spline on $x_i = a + hi$, where $h = (b-a)/n$.

$$B_i(x) = \begin{cases} 0 & \text{for } x < x_{i-2} \\ \dfrac{(x-x_{i-2})^3}{4h^3} & \text{for } x \in [x_{i-2}, x_{i-1}] \\ -\dfrac{3(x-x_{i-1})^3}{4h^3} + \dfrac{3(x-x_{i-1})^2}{4h^2} + \dfrac{3(x-x_{i-1})}{4h} + \dfrac{1}{4} & \text{for } x \in [x_{i-1}, x_i] \\ -\dfrac{3(x_{i+1}-x)^3}{4h^3} + \dfrac{3(x_{i+1}-x)^2}{4h^2} + \dfrac{3(x_{i+1}-x)}{4h} + \dfrac{1}{4} & \text{for } x \in [x_i, x_{i+1}] \\ \dfrac{(x_{i+2}-x)^3}{4h^3} & \text{for } x \in [x_{i+1}, x_{i+2}] \\ 0 & \text{for } x > x_{i+2}. \end{cases}$$