Numerical Analysis

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with thanks to Endre Süli

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Recap on Cubic Splines

Proposition

The set of cubic splines on a given set of knots $x_0 < x_1 < \cdots < x_n$ is a vector space.

Given nodes $x_0, x_1, \ldots, x_n$ and function values $f_0, f_1, \ldots, f_n$, the unique interpolating natural spline $s(x)$ that satisfies $s(x_i) = f_i$ $(i = 0, \ldots, n)$ can be constructed as

$$s(x) = \sum_{i=0}^{n} f(x_i) C_i(x),$$

where $C_0, \ldots, C_i$ are the *cardinal spline* basis of the vector space of natural cubic splines on the knots $x_0, \ldots, x_n$, defined by

$$C_i(x_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Using the basis of **B-splines** is better, because B-splines are locally defined by the requirement that B_i is a cubic spline with knots $x_0, \ldots, x_n$ and

$$B_i(x_i) = 1 \quad \text{for } i = 2, 3, \ldots, n-2,$$
$$B_i(x) \equiv 0, \quad \text{for } x \notin (x_{i-2}, x_{i+2}).$$

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Each B_i consists of 4 patches of cubic polynomials on the intervals $[x_{i-2}, x_{i-1}]$, $[x_{i-1}, x_i]$, $[x_i, x_{i+1}]$ and $[x_{i+1}, x_{i+2}]$, uniquely defined by

- $0 = s(x_{i-2}) = s'(x_{i-2}) = s''(x_{i-2})$,
- continuity of $s(x)$, $s'(x)$ at x_{i-1},
- $s(x_i) = 1$, $s'(x_i) = 0$,
- symmetry on $[x_i, x_{i+2}]$.

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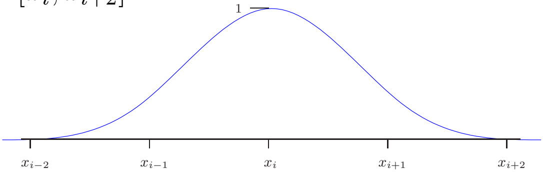

Construction on uniformly spaced knots:

B-spline on $x_i = a + hi$, where $h = (b-a)/n$.

$$B_i(x) = \begin{cases} 0 & \text{for } x < x_{i-2}, \\ \frac{(x-x_{i-2})^3}{4h^3} & \text{for} x \in [x_{i-2}, x_{i-1}], \\ -\frac{3(x-x_{i-1})^3}{4h^3} + \frac{3(x-x_{i-1})^2}{4h^2} + \frac{3(x-x_{i-1})}{4h} + \frac{1}{4} & \text{for} x \in [x_{i-1}, x_i], \\ -\frac{3(x_{i+1}-x)^3}{4h^3} + \frac{3(x_{i+1}-x)^2}{4h^2} + \frac{3(x_{i+1}-x)}{4h} + \frac{1}{4} & \text{for } x \in [x_i, x_{i+1}], \\ \frac{(x_{i+2}-x)^3}{4h^3} & \text{for } \quad x \in [x_{i+1}, x_{i+} \\ 0 & \text{for } x > x_{i+2}. \end{cases}$$

The ‘end’ B-splines B_0, B_1, B_{n-1} and B_n are defined analogously by introducing ‘phantom’ knots $x_{-2} = a - 2h$, $x_{-1} = a - h$, $x_{n+1} = b + h$ and $x_{n+2} = b + 2h$.

Spline interpolation: find the cubic spline

$$s(x) = \sum_{j=0}^{n} c_j B_j(x),$$

that interpolates f_i at x_i for $i = 0, 1, \ldots, n$. Require

$$f_i = \sum_{j=0}^{n} c_j B_j(x_i) = c_{i-1} B_{i-1}(x_i) + c_i B_i(x_i) + c_{i+1} B_{i+1}(x_i).$$

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For equally-spaced data

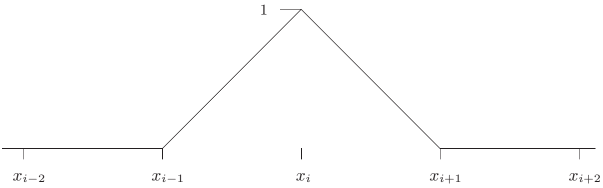
$$f_i = \tfrac{1}{4} c_{i-1} + c_i + \tfrac{1}{4} c_{i+1},$$

i.e.,

$$\begin{pmatrix} 1 & \frac{1}{4} & & & \\ \frac{1}{4} & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1 & \frac{1}{4} \\ & & & \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}.$$

For linear splines, a similar local basis of 'hat functions' or **Linear B-splines** $\phi_i(x)$ exist:

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in (x_{i-1}, x_i) \\ \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & x \in (x_i, x_{i+1}) \\ 0 & x \notin (x_{i-1}, x_{i+1}) \end{cases}$$

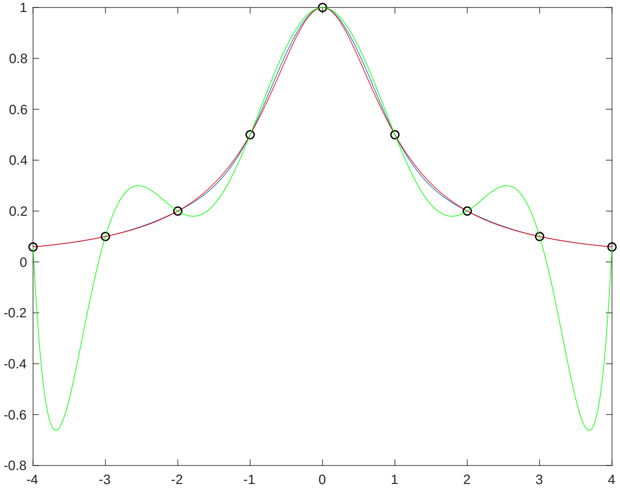


and provide a C^0 piecewise basis.

Matlab:

```
% needs the number N of interpolation points to be defined
N=9;

% these will be the knot points and the interpolation points
x=linspace(-4,4,N);

% a vector of the values of the function f=1/(1+x^2) at the interpolation points
ypoints=1./(1+x.^2);
y=[8/(17^2),ypoints,-8/(17^2)];

% calculates the spline interpolant: see help spline
% an extended vector to include the slope at the first and last
% interpolation point - this is one of the end-point choices available with
% the matlab command spline (and is what is called option (a) in lectures)
% (f' = -2x/(1+x^2)^2, so f'(-4) = 8/17^2 and f'(4) = -8/17^2)
s=spline(x,y);

% a fine mesh on which we plot f
fine=linspace(-4,4,200);

%> help ppval
%
% PPVAL  Evaluate piecewise polynomial.
%    V = PPVAL(PP,XX) returns the value at the points XX of the piecewise
%    polynomial contained in PP, as constructed by SPLINE or the spline utility
%    MKPP.
%
%    See also SPLINE, MKPP, UNMKPP.
plot(fine,ppval(s,fine)),pause

% the function f on the fine mesh
```

```
f=1./(1+fine.^2);

% to see the function (in red) and the spline interpolant (in blue) on the
% same figure
hold on
plot(fine,f,'r'),pause

% marks the interpolating values (with black circles)
plot(x,ypoints,'ko'),pause

% To see how the Lagrange interpolating polynomial (in green) does:
p=lagrange(x,ypoints);
plot(fine,polyval(p,fine),'g'),pause
```


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-4
-3
-2
-1
0
1
2
3
4

Theorem (Smooth Interpolation)

Amongst all functions $t \in \mathrm{C}^2[x_0, x_n]$ *that interpolate* f *at the knots* x_i, $i = 0, 1, \ldots, n$, *the unique function that minimizes*

$$\int_{x_0}^{x_n} [t''(x)]^2 \, \mathrm{d}x$$

is the natural cubic spline s. *Moreover, for any such* t,

$$\int_{x_0}^{x_n} [t''(x)]^2 \, \mathrm{d}x - \int_{x_0}^{x_n} [s''(x)]^2 \, \mathrm{d}x = \int_{x_0}^{x_n} [t''(x) - s''(x)]^2 \, \mathrm{d}x.$$

Error analysis for cubic splines

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Proof. See exercises (uses integration by parts and telescopic cancellation, and is similar to the proof of existence above). □

Lemma (Cauchy–Schwarz Inequality)

Let $f, g \in \mathrm{C}([a, b])$*; then,*

$$\left[\int_a^b f(x)g(x)\,\mathrm{d}x\right]^2 \le \int_a^b [f(x)]^2\,\mathrm{d}x \;\times\; \int_a^b [g(x)]^2\,\mathrm{d}x.$$

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Proof. Since $f^2 \in \mathrm{C}([a,b])$, we have $\int_a^b f^2(x)\,\mathrm{d}x < \infty$, and hence, $f \in L_1^2(a,b)$, the 2-norm space with weight function $w(x) \equiv 1$.

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$$\langle f, g\rangle \leq \|f\| \times \|g\|,$$

see Lecture 9. Taking squares on both sides yields the claim. □

Theorem

For the natural cubic spline interpolant s *of* $f \in \mathrm{C}^2[x_0, x_n]$ *at* $x_0 < x_1 < \cdots < x_n$ *with* $h = \max_{1 \le i \le n}(x_i - x_{i-1})$, *we have that*

$$\|f' - s'\|_\infty \le h^{\frac{1}{2}} \left[\int_{x_0}^{x_n} [f''(x)]^2 \, \mathrm{d}x \right]^{\frac{1}{2}}$$

and

$$\|f - s\|_\infty \le h^{\frac{3}{2}} \left[\int_{x_0}^{x_n} [f''(x)]^2 \, \mathrm{d}x \right]^{\frac{1}{2}}.$$

Proof. Write $e = f - s$. Take any $x \in [x_0, x_n]$, in which case $x \in [x_{j-1}, x_j]$ for some $j \in 1, \ldots, n$.

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$$\begin{aligned}\left(e'(x)\right)^2 &= \left[\int_c^x e''(t)\,\mathrm{d}t\right]^2 \\ &\leq \left|\int_c^x 1\mathrm{d}t\right| \times \left|\int_c^x [e''(t)]^2\,\mathrm{d}t\right|. \end{aligned} \tag{14.1}$$

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Since $x \in [x_{j-1}, x_j]$, we have $\left|\displaystyle\int_c^x 1\mathrm{d}t\right| \le h$, Proof. Write $e = f - s$. Take any $x \in [x_0, x_n]$, in which case $x \in [x_{j-1}, x_j]$ for some $j \in 1, \ldots, n$. Then $e(x_{j-1}) = 0 = e(x_j)$ as s interpolates f. So by the Mean-Value Theorem, there is a $c \in (x_{j-1}, x_j)$ with $e'(c) = 0$. Hence $e'(x) = \int_c^x e''(t)\,\mathrm{d}t$. Then the Cauchy–Schwarz inequality gives that

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Since $x \in [x_{j-1}, x_j]$, we have $\left|\int_c^x 1\mathrm{d}t\right| \le h$, and Theorem (Smooth Interpolation) gives

$$\left|\int_c^x [e''(t)]^2\,\mathrm{d}t\right| \le \int_{x_0}^{x_n} [e''(t)]^2\,\mathrm{d}t \le \int_{x_0}^{x_n} [f''(x)]^2\,\mathrm{d}x.$$

Therefore, $\|f' - s'\|_\infty = \max_{x\in[x_0,x_n]} |e'(x)| \overset{(14.1)}{\le} h^{\frac{1}{2}} \|f''\|_2$, as claimed.

To prove the second claim, still using $x \in (x_{j-1}, x_j)$, Taylor's Theorem yields that there exists $\eta(x) \in (x_{j-1}, x)$ such that

$$|e(x)| = \left| e(x_{j-1}) + (x - x_{j-1}) e'(\eta(x)) \right| \leq 0 + h \left| e'(\eta(x)) \right|,$$

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hence

$$\|f - s\|_\infty = \max_{x\in[x_0,x_n]} |e(x)| \leq h\|e'\|_\infty = h^{\frac{3}{2}}\|f''\|_2,$$

as claimed. □

Recall from Lecture 13:

Theorem

Let s *be the linear spline interpolation of a function* $f \in C^2[a,b]$ *at nodes* $x_0 < x_1 < \cdots < x_n$. *Then,*

$$\|f - s\|_\infty \leq \frac{1}{8} h^2 \|f''\|_\infty.$$

Recall from Lecture 13:

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Is it possible to prove a similar result for cubic splines?

Theorem

Suppose that $f \in \mathrm{C}^4[a, b]$ *and* s *satisfies end-conditions (a). Then,*

$$\|f - s\|_\infty \leq \frac{5}{384}\, h^4 \|f^{(4)}\|_\infty$$

and

$$\|f' - s'\|_\infty \leq \frac{9 + \sqrt{3}}{216}\, h^3 \|f^{(4)}\|_\infty,$$

where $h = \max_{1 \leq i \leq n}(x_i - x_{i-1})$.

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Proof. Beyond the scope of this course. □

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Proof. Beyond the scope of this course. □

Similar bounds exist for natural cubic splines and splines satisfying end-condition (c).