Numerical Analysis Hilary Term 2018
Lecture 10: Best Approximation in Inner-Product Spaces

Best approximation of functions: given a function f on $[a, b]$, find the "closest" polynomial/piecewise polynomial (see later sections)/ trigonometric polynomial (truncated Fourier series).

Norms: are used to measure the size of/distance between elements of a vector space. Given a vector space V over the field $\mathbb{R}$ of real numbers, the mapping $\|\cdot\| : V \to \mathbb{R}$ is a **norm** on V if it satisfies the following axioms:
(i) $\|f\| \geq 0$ for all $f \in V$, with $\|f\| = 0$ if, and only if, $f = 0 \in V$;
(ii) $\|\lambda f\| = |\lambda|\|f\|$ for all $\lambda \in \mathbb{R}$ and all $f \in V$; and
(iii) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in V$ (the **triangle inequality**).

Examples: **1.** For vectors $x \in \mathbb{R}^n$, with $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$,

$$\|x\| \equiv \|x\|_2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} = \sqrt{x^{\mathrm{T}}x}$$

is the ℓ^2- or vector two-norm.
2. For continuous functions on $[a, b]$,

$$\|f\| \equiv \|f\|_\infty = \max_{x \in [a,b]} |f(x)|$$

is the L^∞- or ∞-norm.
3. For integrable functions on (a, b),

$$\|f\| \equiv \|f\|_1 = \int_a^b |f(x)| \, \mathrm{d}x$$

is the L^1- or one-norm.
4. For functions in

$$V = \mathrm{L}^2_w(a, b) \equiv \{f : [a, b] \to \mathbb{R} \mid \int_a^b w(x)[f(x)]^2 \, \mathrm{d}x < \infty\}$$

for some given **weight** function $w(x) > 0$ (this certainly includes continuous functions on $[a, b]$, and piecewise continuous functions on $[a, b]$ with a finite number of jump-discontinuities),

$$\|f\| \equiv \|f\|_2 = \left(\int_a^b w(x)[f(x)]^2 \, \mathrm{d}x\right)^{\frac{1}{2}}$$

is the L^2- or two-norm—the space $\mathrm{L}^2(a, b)$ is a common abbreviation for $\mathrm{L}^2_w(a, b)$ for the case $w(x) \equiv 1$.
Note: $\|f\|_2 = 0 \implies f = 0$ almost everywhere on $[a, b]$. We say that a certain property P holds *almost everywhere* (a.e.) on $[a, b]$ if property P holds at each point of $[a, b]$ except perhaps on a subset $S \subset [a, b]$ of zero measure. We say that a set $S \subset \mathbb{R}$ has *zero measure* (or that it is of *measure zero*) if for any $\varepsilon > 0$ there exists a sequence $\{(\alpha_i, \beta_i)\}_{i=1}^\infty$ of subintervals of $\mathbb{R}$ such that

$S \subset \cup_{i=1}^{\infty}(\alpha_i, \beta_i)$ and $\sum_{i=1}^{\infty}(\beta_i - \alpha_i) < \varepsilon$. Trivially, the empty set $\emptyset(\subset \mathbb{R})$ has zero measure. Any finite subset of $\mathbb{R}$ has zero measure. Any countable subset of $\mathbb{R}$, such as the set of all natural numbers $\mathbb{N}$, the set of all integers $\mathbb{Z}$, or the set of all rational numbers $\mathbb{Q}$, is of measure zero.

Least-squares polynomial approximation: aim to find the best polynomial approximation to $f \in \mathrm{L}^2_w(a,b)$, i.e., find $p_n \in \Pi_n$ for which

$$\|f - p_n\|_2 \leq \|f - q\|_2 \qquad \forall q \in \Pi_n.$$

Seeking p_n in the form $p_n(x) = \displaystyle\sum_{k=0}^{n} \alpha_k x^k$ then results in the minimization problem

$$\min_{(\alpha_0,\ldots,\alpha_n)} \int_a^b w(x) \left[f(x) - \sum_{k=0}^{n} \alpha_k x^k \right]^2 \mathrm{d}x.$$

The unique minimizer can be found from the (linear) system

$$\frac{\partial}{\partial \alpha_j} \int_a^b w(x) \left[f(x) - \sum_{k=0}^{n} \alpha_k x^k \right]^2 \mathrm{d}x = 0 \ \text{ for each } \ j = 0, 1, \ldots, n,$$

but there is important additional structure here.

Inner-product spaces: a real **inner-product space** is a vector space V over $\mathbb{R}$ with a mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ (the **inner product**) for which

(i) $\langle v, v \rangle \geq 0$ for all $v \in V$ and $\langle v, v \rangle = 0$ if, and only if $v = 0$;

(ii) $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$; and

(iii) $\langle \alpha u + \beta v, z \rangle = \alpha \langle u, z \rangle + \beta \langle v, z \rangle$ for all $u, v, z \in V$ and all $\alpha, \beta \in \mathbb{R}$.

Examples: 1. $V = \mathbb{R}^n$,

$$\langle x, y \rangle = x^{\mathrm{T}} y = \sum_{i=1}^{n} x_i y_i,$$

where $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, \ldots, y_n)^{\mathrm{T}}$.

2. $V = \mathrm{L}^2_w(a,b) = \{f : (a,b) \to \mathbb{R} \mid \displaystyle\int_a^b w(x)[f(x)]^2 \, \mathrm{d}x < \infty\}$,

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x) \, \mathrm{d}x,$$

where $f, g \in \mathrm{L}^2_w(a,b)$ and w is a weight-function, defined, positive and integrable on (a,b).

Notes: 1. Suppose that V is an inner product space, with inner product $\langle \cdot, \cdot \rangle$. Then $\langle v, v \rangle^{\frac{1}{2}}$ defines a norm on V (see the final paragraph on the last page for a proof). In Example 2 above, the norm defined by the inner product is the (weighted) L^2-norm.

2. Suppose that V is an inner product space, with inner product $\langle \cdot, \cdot \rangle$, and let $\| \cdot \|$ denote the norm defined by the inner product via $\|v\| = \langle v, v \rangle^{\frac{1}{2}}$, for $v \in V$. The angle θ between $u, v \in V$ is

$$\theta = \cos^{-1} \left(\frac{\langle u, v \rangle}{\|u\| \|v\|} \right).$$

Thus u and v are orthogonal in $V \iff \langle u, v\rangle = 0$.

E.g., x^2 and $\frac{3}{4} - x$ are orthogonal in $\mathrm{L}^2(0,1)$ with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$ as

$$\int_0^1 x^2\left(\tfrac{3}{4} - x\right)\,\mathrm{d}x = \tfrac{1}{4} - \tfrac{1}{4} = 0.$$

3. Pythagoras Theorem: Suppose that V is an inner-product space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ defined by this inner product. For any $u, v \in V$ such that $\langle u, v\rangle = 0$ we have

$$\|u \pm v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof.

$$\begin{aligned}
\|u \pm v\|^2 &= \langle u \pm v, u \pm v\rangle = \langle u, u \pm v\rangle \pm \langle v, u \pm v\rangle && \text{[axiom (iii)]}\\
&= \langle u, u \pm v\rangle \pm \langle u \pm v, v\rangle && \text{[axiom (ii)]}\\
&= \langle u, u\rangle \pm \langle u, v\rangle \pm \langle u, v\rangle + \langle v, v\rangle\\
&= \langle u, u\rangle + \langle v, v\rangle && \text{[orthogonality]}\\
&= \|u\|^2 + \|v\|^2.
\end{aligned}$$

4. The **Cauchy–Schwarz inequality**: Suppose that V is an inner-product space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ defined by this inner product. For any $u, v \in V$,

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

Proof. For every $\lambda \in \mathbb{R}$,

$$0 \leq \langle u - \lambda v, u - \lambda v\rangle = \|u\|^2 - 2\lambda\langle u, v\rangle + \lambda^2\|v\|^2 = \phi(\lambda),$$

which is a quadratic in λ. The minimizer of ϕ is at $\lambda_* = \langle u, v\rangle/\|v\|^2$, and thus since $\phi(\lambda_*) \geq 0$, $\|u\|^2 - \langle u, v\rangle^2/\|v\|^2 \geq 0$, which gives the required inequality. □

5. The **triangle inequality**: Suppose that V is an inner-product space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ defined by this inner product. For any $u, v \in V$,

$$\|u + v\| \leq \|u\| + \|v\|.$$

Proof. Note that

$$\|u + v\|^2 = \langle u + v, u + v\rangle = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2.$$

Hence, by the Cauchy–Schwarz inequality,

$$\|u + v\|^2 \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2.$$

Taking square-roots yields

$$\|u + v\| \leq \|u\| + \|v\|.$$

□

Note: The function $\|\cdot\| : V \to \mathbb{R}$ defined by $\|v\| := \langle v, v\rangle^{\frac{1}{2}}$ on the inner-product space V, with inner product $\langle\cdot,\cdot\rangle$, trivially satisfies the first two axioms of norm on V; this is a consequence of $\langle\cdot,\cdot\rangle$ being an inner product on V. Result 5 above implies that $\|\cdot\|$ also satisfies the third axiom of norm, the triangle inequality.