C5.5
Perturbation Theory

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1. Introduction

- Perturbation methods exploit a small, or large, parameter to make systematic, precise approximations.

Difficult to give rules for perturbation methods, only guidelines.

- Hinch, Bender & Orzag, Supplementary Notes online.

2. Algebraic Equations

Example $\qquad x^2 + \varepsilon x - 1 = 0, \qquad |\varepsilon| \ll 1.$

$$x = \left[-\frac{\varepsilon}{2} \pm \sqrt{1 + \left(\frac{\varepsilon}{2}\right)^2}\,\right] \overset{\text{binomial expansion}}{=} \begin{cases} 1 - \varepsilon/2 + \varepsilon^2/8 + \dots \\ -1 - \varepsilon/2 - \varepsilon^2/8 + \dots \end{cases}$$

for $|\varepsilon/2| < 1$.
$\underbrace{\phantom{for\ |\varepsilon/2|<1}}_{\text{for convergence}}$

In addition to convergence, truncated expansions give good approximations to the roots when $|\varepsilon| \ll 1$.

For $\varepsilon = 0.1$ and the positive root

$x \doteq$	1	1st term	exact root is 0.95124922...
	0.95	2nd term	
	0.95125	3rd term	
	0.951249	4th term	

Solved, then approximated .... usually we approximate, then solve

2.1 Iterative method

$$x^2 + \varepsilon x - 1 = 0$$

For positive root $x = \sqrt{1 - \varepsilon x}$ by rearrangement.

Consider iteration $x_{n+1} = g_\varepsilon(x_n) := \sqrt{1 - \varepsilon x_n}$

Note, if x^* is a root, so that $x_* = g_\varepsilon(x_*)$ then if $|x_n - x^*|$ is small,

$$x_{n+1} - x^* = g_\varepsilon(x_n) - x^* = g_\varepsilon(x^* + (x_n - x^*)) - x^*$$

$$= \underbrace{(g_\varepsilon(x^*) - x^*)}_{0} + (x_n - x^*) g_\varepsilon'(x^*) + \cdots$$

Also $g_\varepsilon'(x^*) = \dfrac{-\varepsilon/2}{\sqrt{1 - \varepsilon x^*}} \simeq -\varepsilon/2 \quad \therefore |x_{n+1} - x^*| \simeq \left|\dfrac{\varepsilon}{2}\right| |x_n - x^*|$

Hence iteration converges. at least if initial guess x_0 is sufficiently close to x^*

Beginning with $x_0 = 1$, $x_1 = \sqrt{1 - \varepsilon} = \underbrace{1 - \varepsilon/2}_{\text{Correct}} - \varepsilon^2/8 - \varepsilon^3/16 + \cdots$

$$x_2 = \sqrt{1 - \varepsilon\left(1 - \frac{\varepsilon}{2} + \cdots\right)}$$

$$= 1 - \frac{\varepsilon}{2}\left(1 - \varepsilon/2 + \cdots\right) - \frac{\varepsilon^2}{8}\left(1 - \varepsilon/2 + \cdots\right)^2 - \frac{\varepsilon^3}{16}\left(1 - \frac{\varepsilon}{2} + \cdots\right)^3$$

$$= \underbrace{1 - \varepsilon/2 + \varepsilon^2/8}_{\text{Correct}} + \underbrace{\cdots}_{\text{Not correct}}$$

At each iteration, more terms correct, but more work required

If solution not known, can only confirm terms are correct by performing a further iteration and checking they do not change

For fast convergence, ideally want $g_\varepsilon(x)$ such that $g'_\varepsilon(x^*) \to 0$ as $\varepsilon \to 0$.

2.2 Expansion Method $\boxed{\text{(Much more common)}}$

For $\varepsilon = 0$, $x = \pm 1$.

to be determined
no dependence on ε

<u>Positive root</u> Let $x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

$$(1+\varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 + \varepsilon(1+\varepsilon x_1 + \varepsilon^2 x_2 + \dots) - 1 = 0$$

<u>Term involving ε^0</u>

$$1 - 1 = 0$$

ε^1 $\quad 2x_1 + 1 = 0 \quad \therefore x_1 = -\tfrac{1}{2}$

ε^2 $\quad 2x_2 + x_1^2 + x_1 = 0 \quad \therefore x_2 = \tfrac{1}{8}$ etc.

set coefficients of powers of ε to zero for each power... ~~assume~~ as this is to hold for all sufficiently small ε.

<u>Caveat</u> Must know/assume form of expansion

2.3 Singular Perturbations

$$\varepsilon x^2 + x - 1 = 0 \qquad |\varepsilon| \ll 1$$

$\varepsilon = 0$, one root $x = 1$. $\varepsilon \neq 0$ two roots.

<u>Singular</u> ... the case with $\varepsilon = 0$ differs in an important way from the case with $\varepsilon \to 0$.

Non-singular problems are <u>regular</u>.

Solve $$x = \frac{1}{2\varepsilon}\left[-1 \pm \sqrt{1+4\varepsilon}\right]$$

$$= \begin{cases} 1-\varepsilon+2\varepsilon^2+\dots \\ -\frac{1}{\varepsilon}-1+\varepsilon-2\varepsilon^2+\dots \end{cases}$$ for $|4\varepsilon|<1$ by binomial expansion

Second root blows up as $\varepsilon \to 0$.

Iterative method

$g_\varepsilon(x) = 1-\varepsilon x^2$ for 1^{st} root

$g_\varepsilon(x) = \frac{1-x}{\varepsilon x}$ for 2^{nd} root

Both derivatives are small near their respective roots and tend to zero as $\varepsilon \to 0$.

Expansion method (2^{nd} root)

2^{nd} root differs from above example

Let $x = \frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ and consider $\varepsilon x^2 + x - 1 = 0$

At ε^{-1} $\quad x_{-1}^2 + x_{-1} = 0$

At ε^{0} $\quad 2x_{-1}x_0 + x_0 - 1 = 0$

At ε^{1} $\quad (2x_{-1}x_1 + x_0^2) + x_1 = 0$

$x_{-1} = 0, -1$

$x_0 = 1, -1$

$x_1 = -1, 1$

regular root (first column) $\quad$ 2^{nd} root (second column)

Rescaling

Let $x = X/\varepsilon \Rightarrow X^2 + X - \varepsilon = 0$, regular.

Finding correct starting point for expansion same as finding a rescaling that makes problem regular.

rescaling that makes problem regular

2.4 Finding the correct rescaling

Systematic approach $\quad x = \delta(\varepsilon) X$ with X strictly order 1 as $\varepsilon \to 0$

We have

$$\varepsilon \delta^2 X^2 + \delta X - 1 = 0$$
$$\quad ① \qquad ② \quad ③$$

Vary δ from very small to large to identify dominant balances, where at least 2 terms are of the same order of magnitude, with all other terms smaller.

Scalings that yield dominant balances are distinguished limits.

$\delta \ll 1$	$① \ll ② \ll ③$	No balance
$\delta = 1$	$① \ll ② \sim ③$ (same order of magnitude)	Balance, regular root $X = 1 + \text{small}$
$1 \ll \delta \ll 1/\varepsilon$	$① \ll ② \gg ③$	No balance
$\delta = 1/\varepsilon$	$① \sim ② \gg ③$	Balance, singular root $x = 1/\varepsilon + \ldots$ $X = -1 + \text{small}$
$\delta \gg 1/\varepsilon$	$① \gg ② \gg ③$	No balance

$\therefore$ Distinguished limits are $\delta = 1, 1/\varepsilon$.

Alternative approach: Pairwise comparison

1 ~ 2 $\quad \varepsilon\delta^2 \sim \delta$ i.e. $\delta \sim 1/\varepsilon$ and 1 ~ 2 ≫ 3 ∴ Singular root.

1 ~ 3 $\quad \varepsilon\delta^2 \sim 1$ i.e. $\delta \sim 1/\sqrt{\varepsilon}$ and 1 ~ 3 ≪ 2 ∴ No dominant balance

2 ~ 3 $\quad \delta \sim 1$ $\quad$ 2 ~ 3 ≫ 1 ∴ Regular root.

2.5 Non-integer powers

$$(1-\varepsilon)x^2 - 2x + 1 = 0 \qquad |\varepsilon| \ll 1$$

$$x = \frac{1 \pm \sqrt{\varepsilon}}{1-\varepsilon} = 1 \pm \sqrt{\varepsilon} + \varepsilon \pm \varepsilon^{3/2} + \dots$$

double root ... sign of danger

With $\varepsilon = 0$ $\quad (x-1)^2 = 0 \quad \therefore x = 1.$

Try $x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ $\qquad$ Know this will go wrong

Objective is to see how it goes wrong so we know what to do if we see analogous behaviour in the expansion method when we do not know the solution

At ε^0 $\qquad 1 - 2 + 1 = 0$

At ε^1 $\qquad -1 + 2x_1 - 2x_1 = 0$ $\qquad$ No solution, "unless x_1 blows up in some sense".

Try $x = 1 + \varepsilon^{1/2} x_{1/2} + \varepsilon x_1 + \varepsilon^{3/2} x_{3/2} + \dots$

At ε^0 $\qquad 1 - 2 + 1 = 0$

At $\varepsilon^{1/2}$ $\qquad 2x_{1/2} - 2x_{1/2} = 0$

At ε $\qquad 2x_1 + x_{1/2}^2 - 1 - 2x_1 = x_{1/2}^2 - 1 = 0$

$$\therefore x_{1/2} = \pm 1$$

etc.

2.6 Finding the correct expansion sequence

Let $x = 1 + \delta_1(\varepsilon) x_1$ where $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, with x_1 strictly order one.

root when $\varepsilon = 0$

$$(1-\varepsilon)(1+\delta_1 x_1)^2 - 2(1+\delta_1 x_1) + 1 = 0$$

$$\cancel{1} + \cancel{2\delta_1 x_1} + \delta_1^2 x_1^2 - \varepsilon(1+2\delta_1 x_1 + \delta_1^2 x_1^2) \cancel{-2} \cancel{-2\delta_1 x_1} + \cancel{1} = 0$$

$$\underset{①}{\delta_1^2 x_1^2} - \underset{②}{\varepsilon} - \underset{③}{2\varepsilon\delta_1 x_1} - \underset{④}{\varepsilon\delta_1^2 x_1^2} = 0 \qquad (*)$$

Seek dominant balance   ④ $\ll$ ① always   ③ $\ll$ ② always

$\therefore$ ④③ not in dominant balance $\therefore$ ① $\sim$ ② $\therefore$ $\underline{\underline{\varepsilon \sim \delta_1^2}}$

$\therefore$ Let $\delta_1 = \sqrt{\varepsilon}$

$\therefore$ $\left.\begin{array}{l} x = 1 + \varepsilon^{1/2} x_1 + \delta_2(\varepsilon) x_1 + \cdots \\ \quad \text{with } \delta_2(\varepsilon) \ll \delta_1 = \varepsilon^{1/2} \end{array}\right\}$ From $(*)$ we get at $O(\varepsilon)$

$x_1^2 - 1 = 0 \quad \therefore \underline{\underline{x_1 = \pm 1}}$

With root with $x_1 = 1$

$\therefore$ $x = 1 + \varepsilon^{1/2} + \delta_2(\varepsilon) x_2 + \cdots$

$$(1-\varepsilon)(1+\varepsilon^{1/2} + \delta_2 x_2 + \cdots)^2 - 2(1+\varepsilon^{1/2} + \delta_2 x_2 + \cdots) + 1 = 0$$

Lots of algebra retaining all terms

$$\underset{①}{2\varepsilon^{1/2}\delta_2 x_2} + \underset{②}{\delta_2^2 x_2^2} - \underset{③}{2\varepsilon^{3/2}} - \underset{④}{\varepsilon^2} - \underset{⑤}{2\varepsilon\delta_2 x_2}$$
$$- \underset{⑥}{2\varepsilon^{3/2}\delta_2 x_2} - \underset{⑦}{\varepsilon\delta_2^2 x_2^2} + \cdots = 0$$

Want dominant terms, but only know $\delta_2 \ll \varepsilon^{1/2}$

$x_2 \sim$ order one

4 $\ll$ 3 for terms with no δ_2.

For terms with δ_2,

$$① \gg ⑦,\quad ① \sim \varepsilon^{1/2}\delta_2 \qquad ⑦ \sim \varepsilon\delta_2^2 = (\varepsilon^{1/2}\delta_2)(\varepsilon^{1/2}\delta_2)$$
$$\sim ①\, \varepsilon^{1/2}\delta_2 \ll ①$$

Similarly for all other terms

$\therefore$ Dominant balance is between 1 and 3

$$2\varepsilon^{1/2}\delta_2 x_2 - 2\varepsilon^{3/2} = 0$$

$$\therefore \quad \delta_2 x_2 = \varepsilon \qquad \text{Let } \delta_2 = \varepsilon,\ x_2 = 1$$

$$\therefore \quad x = 1 + \varepsilon^{1/2} + \varepsilon + \dots$$

2.7 Iterative Method (again)

· Useful when expansion form not known

$$(1-\varepsilon)x^2 - 2x + 1 = 0$$
$$(x-1)^2 = \varepsilon x^2$$

For root > 1

$$x_{n+1} = g_\varepsilon(x_n) = 1 + \varepsilon^{1/2} x_n$$

Note $g'_\varepsilon(x_n) \to 0$ as $\varepsilon \to 0$

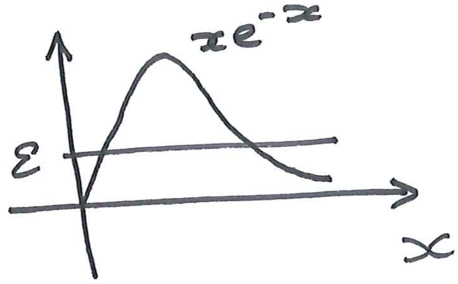
$\therefore x_0 = 1$ $\quad x_1 = 1 + \varepsilon^{1/2},\ x_2 = \dots$ etc

solution if $\varepsilon = 0$

generates sequence

2.8 Logarithms

Example $\quad xe^{-x} = \varepsilon \qquad 0<\varepsilon\ll 1$



Root near $x=0$ easy to find.

$$\left[\text{Let } x = \underset{\substack{\text{solution}\\ \text{if } \varepsilon=0}}{0} + \varepsilon x_1 + \varepsilon^2 x_2 + \dots\dots \right]$$

Taylor expand $x\exp(-x)$ about zero ... generates integer powers of x and hence integer powers of εx_1, $\varepsilon^2 x_2$, etc, Thus a series constructed from a sequence of integerpowers of ε will enable a balance. Hence the form of the series.

Other root $\to \infty$ as $\varepsilon \to 0$

Hence the form of the expansion to find approximate solutions is not immediately clear

Take logs

$$\underset{①}{x} - \underset{②}{\log x} - \underset{③}{\log 1/\varepsilon} = 0$$

For x large $|①| \gg |②|$ $\therefore$ ② not in dominant balance

$\therefore \quad x \sim \log 1/\varepsilon \quad$ as $\varepsilon \to 0^+$.

Suggests $\qquad x_{n+1} = g_\varepsilon(x_n) = \log(x_n) + \log(1/\varepsilon)$

Note $g_\varepsilon'(x) = 1/x$

$$g_\varepsilon'(x^*) \sim \frac{1}{\log(1/\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0^+$$

[but slow convergence]

$\therefore \quad x_0 = \log \frac{1}{\varepsilon}$

$$x_1 = g_\varepsilon(x_0) = g_\varepsilon\left(\log \tfrac{1}{\varepsilon}\right) = \log\left(\tfrac{1}{\varepsilon}\right) + \log\left(\log\left(\tfrac{1}{\varepsilon}\right)\right)$$

$$x_2 = g_\varepsilon\left(\log \tfrac{1}{\varepsilon} + \log\left(\log \tfrac{1}{\varepsilon}\right)\right)$$

$$= \log \tfrac{1}{\varepsilon} + \log\left(\log \tfrac{1}{\varepsilon}\left(1 + \frac{\log(\log \frac{1}{\varepsilon})}{\log \frac{1}{\varepsilon}}\right)\right)$$

$\log(1+\delta) \simeq \delta$ for $|\delta| < 1$

$$= \log \tfrac{1}{\varepsilon} + \log\left(\log \tfrac{1}{\varepsilon}\right) + \frac{\log(\log(\frac{1}{\varepsilon}))}{\log(\frac{1}{\varepsilon})} + \cdots$$

Don't know answer... need to compute x_3 to confirm first 3 terms are correct.

Difficult sequence to guess.

Converges VERY slowly

3. Asymptotic Approximations

3.1 Definitions

- A series $\sum_{n=0}^{\infty} f_n(z)$ <u>converges</u> at fixed z if $\forall \varepsilon > 0, \exists N_0(z, \varepsilon) \in \mathbb{N}$ such that
$$\left| \sum_{n=m}^{N} f_n(z) \right| < \epsilon \quad \forall N \geq m > N_0$$

- A series $\sum_{n=0}^{\infty} f_n(z)$ converges to $f(z)$ at fixed z if $\forall \varepsilon > 0, \exists N_0(z, \varepsilon) \in \mathbb{N}$ such that
$$\left| f(z) - \sum_{n=0}^{N} f_n(z) \right| < \varepsilon \quad \forall N \geq N_0$$

- A series converges if its terms decay sufficiently rapidly as $n \to \infty$

- Less useful in practice than might be believed.

Example $$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt, \qquad z \in \mathbb{C}.$$

e^{-t^2} is a holomorphic function of $t \in \mathbb{C}$.

Thus it has a convergent power series with infinite radius of convergence

$$\sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}$$

Integrate term by term

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)\,n!} = \frac{2}{\sqrt{\pi}} \left(z - z^3/3 + z^5/10 \cdots\cdots \right)$$

Has infinite radius of convergence.

For accuracy of 10^{-5}, 16 terms needed for $z = 2$
31 terms needed for $z = 3$
75 terms needed for $z = 5$

Cancellation required between large powers... need lot of terms for good approximation

Alternative approach to approximating erf(z).

Rewrite $$\operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\,dt$$

Parts

$$-\tfrac{1}{2t} \qquad -2te^{-t^2}$$
$$\tfrac{1}{2t^2} \qquad e^{-t^2}$$

$$\int_z^\infty e^{-t^2}dt = \int_z^\infty \left(-\frac{1}{2t}\right)\left(-2te^{-t^2}\right)dt$$
$$= \left[-\tfrac{1}{2t}e^{-t^2}\right]_z^\infty - \int_z^\infty \frac{1}{2t^2}e^{-t^2}dt$$
$$= \frac{1}{2z}e^{-z^2} - \int_z^\infty \frac{e^{-t^2}}{2t^2}\,dt$$

Continuing the integration by parts

$$\operatorname{erf}(z) = 1 - \frac{e^{-z^2}}{z\sqrt{\pi}}\left(1 - \frac{1}{2z^2} + \frac{1.3.}{(2z^2)^3} - \frac{1.3.5}{(2z^2)^5} + \cdots\right)$$

This series diverges $\forall z \in \mathbb{C}$, but truncated series very useful.

- For accuracy of 10^{-5} only two terms are needed for $z = 3$.
- Importantly The leading term is almost correct and each additional term gets us closer to the answer, with each additional correction of decreasing size until eventually they start increasing.
- This is an asymptotic series

Asymptoticness

- A sequence $\{f_n(\epsilon)\}_{n \in \mathbb{N}_0}$ is asymptotic if $\forall n \geq 1$

$$\frac{f_n(\varepsilon)}{f_{n-1}(\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0$$

- A series $\sum_{n=0}^{\infty} f_n(\epsilon)$ is an asymptotic expansion of a function $f(\varepsilon)$ as $\varepsilon \to 0$ if $\forall N \in \mathbb{N}_0$
$$\frac{f(\varepsilon) - \sum_{n=0}^{N} f_n(\epsilon)}{f_N(\epsilon)} \to 0 \quad \text{as } \varepsilon \to 0.$$

- In other words, the remainder is smaller than the last term included once ε is sufficiently small.

- We write $f(\varepsilon) \sim \sum_{n=0}^{\infty} f_n(\epsilon)$ as $\varepsilon \to 0$

- Usually first few terms are sufficient for a good approximation

- Often $f_n(\varepsilon) = a_n \varepsilon^n$ with a_n real, in which case
$$f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \varepsilon^n \quad \text{as } \varepsilon \to 0$$
is called an asymptotic power series.

$\left\{ f_n = a_n \delta_n(\varepsilon) \right.$ with $\{\delta_n(\varepsilon)\}_{n \in \mathbb{N}_0}$ asymptotic also common $\left. \right\}$.

Order Notation

- $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means
  $\exists K, \delta > 0$ s.t $|f(\varepsilon)| < K|g(\varepsilon)| \quad \forall\, |\varepsilon - \varepsilon_0| < \delta$

- $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means $\dfrac{f(\varepsilon)}{g(\varepsilon)} \to 0$ as $\varepsilon \to \varepsilon_0$

- $f(\varepsilon) = \mathrm{ord}(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means
  $\exists K \in \mathbb{R}\setminus\{0\}$ s.t. $\dfrac{f(\varepsilon)}{g(\varepsilon)} \to K$ as $\varepsilon \to \varepsilon_0$

Examples

$\sin(x) = O(1),\ o(1),\ O(x),\ \mathrm{ord}(x)$ as $x \to 0$

$\sin(x) = O(1)$ as $x \to \infty$

$\log(x) = o(x^{-\delta})$ as $x \to 0$ for any $\delta > 0$.

3.2 Uniqueness and manipulation of an asymptotic series

- If a function $f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \delta_n(\varepsilon)$ as $\varepsilon \to 0$ then induction implies that

$\{a_n\}_{n \in \mathbb{N}_0}$ is uniquely determined by $a_k = \lim_{\varepsilon \to 0} \left[\dfrac{f(\varepsilon) - \sum_{n=0}^{k-1} a_n \delta_n(\varepsilon)}{\delta_k(\varepsilon)} \right]$

- Uniqueness is for a <u>given</u> sequence $\{\delta_n(\varepsilon)\}_{n \in \mathbb{N}_0}$

- The sequence need not be unique e.g.

$$\tan \varepsilon \sim \varepsilon + \varepsilon^3/3 + 2\varepsilon^5/15 + \cdots \quad \text{as } \varepsilon \to 0$$

$$\tan \varepsilon \sim \sin \varepsilon + \tfrac{1}{2}(\sin \varepsilon)^3 + 3/8\,(\sin \varepsilon)^5 + \cdots \quad \text{as } \varepsilon \to 0$$

- Uniqueness for a <u>given</u> function... two functions may share the same asymptotic expansion e.g.

$$e^{\varepsilon} \sim \sum_{n=0}^{\infty} \varepsilon^n/n! \quad \text{as } \varepsilon \to 0$$

$$e^{\varepsilon} + e^{-1/\varepsilon^2} \sim \sum_{n=0}^{\infty} \varepsilon^n/n! \quad \text{as } \varepsilon \to 0$$

- Two distinct functions with the same asymptotic power series can only differ by a function that is not holomorphic as two holomorphic functions with the same power series are identical.

- Asymptotic expansions can be naively added, subtracted, divided, multiplied and divided (though the sequence eg. the $\{\delta_n(\epsilon)\}_{n\in\mathbb{N}_0}\}$ may be larger).

- This underlies expansion method for algebraic equations.

- One series can be substituted into another, but take care with exponentials .... always expand exponents to ord(1).

<u>Example</u> $f(z) = e^{z^2}$ $\quad z = 1/\varepsilon + \varepsilon$ $\quad$ Naively $f(z) \sim e^{1/\varepsilon^2}$ at leading order $\quad \times$

$$f(z) = \exp\left(\left(\frac{1}{\varepsilon}+\varepsilon\right)^2\right) = \exp\left(\frac{1}{\varepsilon^2}+2+\varepsilon^2\right) = e^{1/\varepsilon^2}\cdot e^2\cdot\left(1+\varepsilon^2+\frac{(\varepsilon^2)^2}{2!}+\frac{(\varepsilon^2)^3}{3!}+\dots\right)$$

- Sine and Cosine and complex exponentials require analogous care in this context.

- Asymptotic expansions can be integrated term by term with respect to ε resulting in the correct asymptotic expansion for the integral.

- In general asymptotic expansions <u>cannot</u> be differentiated safely

<u>Example</u>

$$f(\varepsilon) = \varepsilon \cos(1/\varepsilon) = o(\varepsilon) \text{ as } \varepsilon \to 0$$

$$f'(\varepsilon) = \frac{1}{\varepsilon}\sin(1/\varepsilon) + \cos(1/\varepsilon) = o(1/\varepsilon) \text{ as } \varepsilon \to 0$$

Differentiating the asymptotic expansion with the $o(\varepsilon)$ start would naively give $o(1)$ ... but the derivative is $o(1/\varepsilon)$ as $\varepsilon \to 0$.

- Terms move down an asymptotic expansion with differentiation (eg. $\frac{d}{dx} x^n = nx^{n-1}$) and thus terms at higher orders may cause problems on differentiation.

- Often, first few terms sufficient. If higher accuracy required...

  <u>optimal truncation</u> : truncate asymptotic series at smallest term

3.4 Parametric Expansions

- Integrals, differential equations and partial differential equations involve functions with one, or more, variables $f(x,\varepsilon)$ with ε a small parameter.

- There is an obvious generalisation of the definition of an asymptotic expansion by allowing the coefficients to depend on x. For fixed x

$$f(x;\varepsilon) \sim \sum_{n=0}^{\infty} a_n(x)\,\delta_n(\varepsilon) \quad \text{as } \varepsilon \to 0 \quad \text{if and only if}$$

$$\frac{1}{\delta_N(\varepsilon)}\left[f(x;\varepsilon) - \sum_{n=0}^{N} a_n(x)\,\delta_n(\varepsilon)\right] \to 0 \quad \text{as } \varepsilon \to 0$$

4. Asymptotic expansions of integrals

4.1 Integration by parts

Example derivation of an asymptotic power series

f' differentiable near $\varepsilon = 0$; $f(\varepsilon) = f(0) + \int_0^\varepsilon f'(x)\,dx$

Write $1 = \frac{d}{dx}(x-\varepsilon)$

Parts $f(\varepsilon) = f(0) + \Big[(x-\varepsilon)f'(x)\Big]_0^\varepsilon$

$$- \int_0^\varepsilon (x-\varepsilon) f''(x)\,dx$$

$$= f(0) + \varepsilon f'(0) - \int_0^\varepsilon (x-\varepsilon) f''(x)\,dx$$

Repeat

$$= \sum_{n=0}^{N} \frac{f^{(n)}(0)\,\varepsilon^n}{n!} + \frac{1}{N!}\int_0^\varepsilon (\varepsilon - x)^N f^{(N+1)}(x)\,dx$$

If remainder term exists for $\forall N \in \mathbb{N}$ and sufficiently small ε, then

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,\varepsilon^n}{n!} \quad \text{as } \varepsilon \to 0$$

[If the series converges, it is the Taylor series about zero].

Example $I(x) = \int_x^\infty e^{-t^4}\,dt$

Want asymptotic series as $x \to \infty$

no taylor series !!!!

$$I(x) = \int_x^\infty \left(\frac{-1}{4t^3}\right)\left(-4t^3 e^{-t^4}\right) dt$$

$$= \left[\left(-\frac{1}{4t^3}\right) e^{-t^4}\right]_x^\infty - \int_x^\infty \left(\frac{3}{4t^4}\right) e^{-t^4}\, dt$$

$$= \frac{e^{-x^4}}{4x^3} - \frac{3}{4}\underbrace{\int_x^\infty \frac{e^{-t^4}}{t^4}\, dt} \sim \frac{e^{-x^4}}{4x^3} \text{ as } x \to \infty$$

$$\int_x^\infty \frac{e^{-t^4}}{t^4}\, dt < \frac{1}{x^4} e^{-x^4} \int_x^\infty e^{-(t^4 - x^4)}\, dt$$

$$t^4 - x^4 = (t-x)(t+x)(t^2+x^2)$$

Let $u = t - x$

$$\int_x^\infty e^{-(t^4-x^4)}\, dt = \int_0^\infty e^{-u(u+2x)((u+x)^2+x^2)}\, du$$

$$< \int_0^\infty e^{-u^4}\, du < \int_0^\infty e^{-u^2}\, du$$

$$\therefore \int_x^\infty \frac{e^{-t^4}}{t^4}\, dt \sim O\left(\frac{1}{x^4} e^{-x^4}\right) \ll \frac{e^{-x^4}}{4x^3}$$

Correction much smaller than "last" term

Further integration by parts will give higher order terms.

Example $$I(x) = \int_0^x t^{-1/2} e^{-t}\,dt$$

Naive Integration by parts fails.

$$I(x) = \left[-t^{-1/2} e^{-t}\right]_0^x - \int_0^x \underbrace{\left(-\tfrac{1}{2} t^{-3/2}\right)}_{\text{Not integrable}} \left(-e^{-t}\right) dt$$

divergence at $x=0$ not really an issue ... take care of it separately

$$\therefore I(x) = \underbrace{\int_0^\infty t^{-1/2} e^{-t}\,dt}_{\Gamma(1/2) = \sqrt{\pi}} - \underbrace{\int_x^\infty t^{-1/2} e^{-t}\,dt}_{J(x)}$$

use substitution $u = \sqrt{t}$

$$J(x) = \int_x^\infty t^{-1/2} e^{-t}\,dt = \left[-t^{-1/2} e^{-t}\right]_x^\infty - \frac{1}{2}\int_x^\infty t^{-3/2} e^{-t}\,dt$$

$$= \frac{e^{-x}}{\sqrt{x}} - \frac{1}{2}\underbrace{\int_x^\infty \frac{e^{-t}}{t^{3/2}}\,dt}$$

$$< \frac{1}{x^{3/2}} \int_x^\infty e^{-t}\,dt = \frac{e^{-x}}{x^{3/2}} \ll \frac{e^{-x}}{x^{1/2}}$$

$$\frac{\text{Correction}}{\text{Last term}} \to 0 \text{ as } x \to \infty$$

$$\therefore I(x) \sim \sqrt{\pi} - e^{-x}/\sqrt{x} + \ldots$$

<u>General Rule</u> Integration by parts works if the contribution from one of the limits of the integration dominates

4.2 Failure of Integration by Parts

Example $I(x) = \int_0^\infty e^{-xt^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{x}}$ for $x > 0$.

let $u = x^{1/2} t$

Attempt (Parts)

$$I(x) = \int_0^\infty \left(\frac{-1}{2xt}\right)\left(-2xte^{-xt^2}\right) dt$$

$$= \underbrace{\left[\frac{e^{-xt^2}}{-2xt}\right]_0^\infty - \int_0^\infty \frac{e^{-xt^2}}{2xt^2}\,dt}_{}$$

does not exist; fractional power in x not picked up by this type of expansion

$\therefore$ Integration by parts simple but inflexible, of limited use.

Also does not work when dominant contribution to integral is from domain interior (need one limit to dominate).

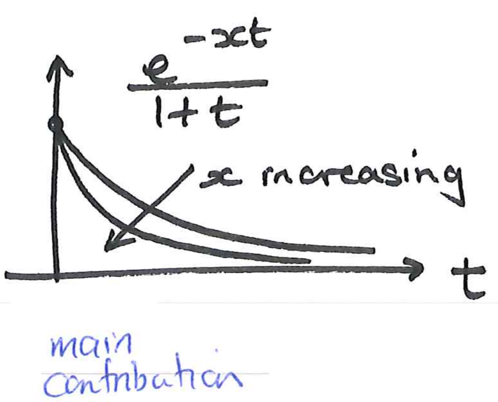
4.3 Laplace's Method

General technique for the asymptotic expansion as $x \to \infty$ of

$$I(x) = \int_a^b f(t)\, e^{x\varphi(t)}\, dt$$

with $[a,b] \subseteq \mathbb{R}$ and f, φ continuous real functions on $[a,b]$.

Example $I(x) = \int_0^1 \frac{e^{-xt}}{1+t}\, dt$



$$I(x) = \underbrace{\int_0^{\varepsilon} \frac{e^{-xt}}{1+t}\,dt}_{I_1(x)} + \underbrace{\int_{\varepsilon}^{1} \frac{e^{-xt}}{1+t}\,dt}_{I_2(x)}$$

with $0 < 1/x \ll \varepsilon \ll 1$.

$$I_1(x) = \frac{1}{x}\int_0^{x\varepsilon} \frac{e^{-s}}{1+s/x}\,ds$$

$s/x < x\varepsilon/x = \varepsilon \ll 1$

$\therefore$ within radius of convergence and expansion uniform.

$$= \frac{1}{x}\int_0^{x\varepsilon} e^{-s}\left(\sum_{n=0}^{\infty}\left(\frac{-s}{x}\right)^n\right) ds$$

$$= \frac{1}{x}\sum_{n=0}^{\infty}\left[\int_0^{x\varepsilon} s^n e^{-s}\,ds\right]\frac{(-1)^n}{x^n}$$

$$\int_0^{x\epsilon} s^n e^{-s}\,ds = \int_0^{\infty} s^n e^{-s}\,ds - \int_{x\epsilon}^{\infty} e^{-s} s^n\,ds = n! - \underbrace{\int_{x\epsilon}^{\infty} s^n e^{-s}\,ds}_{K_n}$$

$$K_n \overset{\text{parts}}{=} \underbrace{(x\epsilon)^n e^{-x\epsilon}}_{\substack{\text{exponentially small}\\ \text{for fixed } n \text{ as } x\epsilon \gg 1.}} + n\int_{x\epsilon}^{\infty} s^{n-1} e^{-s}\,ds = \text{exponentially small} + nK_{n-1}$$

$$\therefore K_n = (n!)\int_0^{\infty} e^{-s}\,ds + \text{exponentially small} = (n!)\,e^{-x\epsilon} + \text{exponentially small} \ll n!$$

$$\therefore I_1 = \frac{1}{x}\sum_{n=0}^{\infty}\left(\int_0^{x\epsilon} s^n e^{-s}\,ds\right)\frac{(-1)^n}{x^n} \sim \sum_{n=0}^{\infty}\frac{(-1)^n n!}{x^{n+1}}$$

as exponentially small terms will always be dominated by a power of $(1/x)$ as $x \to \infty$.

Also $I_2 < \int_{\varepsilon}^{1} e^{-xt}\,dt = \left(\underbrace{e^{-x\epsilon}}_{\substack{\text{already dropped}\\ \text{terms this small}}} - e^{-x}\right)/x$ even smaller

$\ll I_1(x)$

$$\therefore I(x) \sim \sum_{n=0}^{\infty}\frac{(-1)^n n!}{x^{n+1}} \text{ as } x \to \infty$$

4.4 Watson's Lemma

Let $$I(x) = \int_0^b f(t) e^{-xt}\, dt \quad , \quad b>0,$$

with (i) $f(t)$ continuous on $t \in [0,b]$

(ii) If $b=\infty$, in addition $\exists c \in \mathbb{R}$ with $f(t) = o(e^{ct})$ as $t \to \infty$

(iii) $$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n} \quad \text{as } t \to 0^+$$

with $\alpha > -1$, $\beta > 0$, $a_n \in \mathbb{R}$ for $n \in \mathbb{N}_0$.

Then $$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \quad \text{as } x \to \infty$$

where $$\Gamma(m) = \int_0^{\infty} t^{m-1} e^{-t}\, dt.$$

Note $\Gamma(m) = (m-1)!$ for $m \in \mathbb{N}$.

<u>Proof</u> See Supplementary Notes online.

$\left\{ \text{If } f \text{ uniformly convergent in neighbourhood of origin, proceeds as in example above} \right.$

4.5 General Laplace Integrals

- Dominant contribution to

$$I(x) = \int_a^b f(t) e^{x\varphi(t)}\, dt \quad \text{as } x \to \infty$$

is from the region where $\varphi(t)$ is the largest.

- There are 3 cases: the maximum of $\varphi(t)$ is at

(i) $t = a$, (ii) $t = b$, (iii) $t = c \in (a,b)$.

To proceed

- Isolate dominant contribution from near maximum of φ and reduce range of integration to this region
  - Gives exponentially small errors
- Taylor expand φ, f and rescale
- Finally extend range of integration once other approximations made

need to assess size of ε relative to $1/x$ ... later!

<u>Case (i)</u> with $\varphi'(a) < 0$, $f(a) \neq 0$, $\varphi''(a) \neq 0$

$$I(x) = \underbrace{\int_a^{a+\varepsilon} f(t) e^{x\varphi(t)}\, dt}_{I_1(x)} + \underbrace{\int_{a+\varepsilon}^{b} f(t) e^{x\varphi(t)}\, dt}_{I_2(x)}$$

$|I_1| \gg |I_2|$

$$e^{x\varphi(a+\varepsilon)} \ll e^{x\varphi(a)} \qquad \varphi(a+\varepsilon) \simeq \varphi(a) + \varepsilon\varphi'(a)$$

$$e^{x\varepsilon\varphi'(a)} \ll 1$$

$$\boxed{x\varepsilon \gg 1}$$

$$I_1(x) = \int_a^{a+\varepsilon} [f(a) + (t-a)f'(a) + \dots] \exp\left[x\left\{\varphi(a) + (t-a)\varphi'(a) + \frac{(t-a)^2}{2}\varphi''(a) + \dots\right\}\right] dt$$

$$= e^{x\varphi(a)} \int_a^{a+\varepsilon} [f(a) + (t-a)f'(a) + \dots] e^{x(t-a)\varphi'(a)} \left[1 + x\frac{(t-a)^2}{2}\varphi''(a) + \dots\right] dt$$

$\boxed{\text{Rescale } x(t-a) = s}$ ← Remove x from leading exponent.

okay given $\boxed{x\varepsilon^2 \ll 1}$ $\therefore$ $\boxed{\frac{1}{x} \ll \varepsilon \ll \frac{1}{\sqrt{x}}}$

$$= \frac{e^{x\varphi(a)}}{x} \int_0^{\varepsilon x} [f(a) + O(s/x)] e^{s\varphi'(a)} \left[1 + O\left(s^2/x\right)\right] ds$$

$\to \infty$ OKAY as $\varepsilon x \gg 1$

Explain in detail

$$= \frac{f(a)e^{x\varphi(a)}}{x} \left(\int_0^{\varepsilon x} e^{s\varphi'(a)} \left(1 + O\left(1/x\right)\right)\right) ds$$

$$= \frac{f(a)e^{x\varphi(a)}}{x|\varphi'(a)|} \left(1 + O\left(1/x\right)\right)$$

← guarantees asymptoticness... correction much smaller than last term.

$$\therefore I(x) \sim I_1(x) \sim \frac{f(a)e^{x\varphi(a)}}{x|\varphi'(a)|} \quad \text{as } x \to \infty.$$

<u>Case (ii)</u> with $\varphi'(b) > 0$, $f(b) \neq 0$, $\varphi''(b) \neq 0$. <u>Exercise</u> Show that

$$I(x) \sim \frac{f(b)e^{x\varphi(b)}}{x\,\varphi'(b)} \quad \text{as } x \to \infty.$$

← Essentially identical to case (i)

Case (iii) $\varphi'(c)=0,\ \varphi''(c)<0,\ f(c)\neq 0,\ \varphi'''(c)\neq 0$

$t=c$ global maximum of $\varphi(t)$ for $t\in[a,b]$.

$$I(x)=\Big[\underbrace{\int_a^{c-\varepsilon}dt}_{I_1}+\underbrace{\int_{c-\varepsilon}^{c+\varepsilon}dt}_{I_2}+\underbrace{\int_{c+\varepsilon}^{b}}_{I_3}\Big]\ f(t)e^{x\varphi(t)}$$

I_2 dominant

$$e^{x\varphi(c+\varepsilon)}\ll e^{x\varphi(c)} \quad \text{for } |I_2|\gg|I_3|$$

$\varphi(c+\varepsilon)\approx\varphi(c)+\frac{\varepsilon^2}{2}\varphi''(c)$ as $\varphi'(c)=0$

$$\therefore \quad e^{\frac{x\varepsilon^2\varphi''(c)}{2}}\ll 1 \quad \therefore \quad \boxed{x\varepsilon^2\gg 1}$$

Same argument for $|I_2|\gg|I_1|$

$$I_2(x)=\int_{c-\varepsilon}^{c+\varepsilon}dt\ f(t)e^{x\varphi(t)}$$

$$=\int_{c-\varepsilon}^{c+\varepsilon}\big[f(c)+O(t-c)\big]e^{x\varphi(c)}\,e^{x(t-c)^2/2\,\varphi''(c)}\cdot\Big[1+\underbrace{O(x(t-c)^3/3!)}_{x\varepsilon^3\ll 1}\Big]dt$$

$$\therefore \quad \underbrace{\boxed{\frac{1}{x^{1/2}}\ll\varepsilon\ll\frac{1}{x^{1/3}}}}_{\text{Need } x \text{ rather large}}$$

eg suppose $x=8$

$\frac{1}{2\sqrt{2}}\ll\varepsilon\ll 1/2$

but $\frac{1}{\sqrt{2}}\not\ll 1$

Rescale $s = \sqrt{x}(t-c)$

$$I_2(x) = \frac{f(c)e^{x\varphi(c)}}{\sqrt{x}} \int_{-\sqrt{x}\varepsilon}^{\sqrt{x}\varepsilon} ds \; e^{s^2/2\,\varphi''(c)} \underbrace{\left(1+O\left(\frac{s}{\sqrt{x}}\right)\right)}_{\text{from expansion of } f} \underbrace{\left(1+O\left(\frac{s^3}{\sqrt{x}}\right)\right)}_{\text{from exponential expansion}}$$

$$= \frac{f(c)e^{x\varphi(c)}}{\sqrt{x}} \underbrace{\int_{-\infty}^{\infty} ds\, e^{s^2/2\,\varphi''(c)}}_{\sqrt{\frac{2}{-\varphi''(c)}}\int_{-\infty}^{\infty} du\, e^{-u^2}} \left(1+O\left(\frac{1}{\sqrt{x}}\right)\right)$$

okay as $\sqrt{x}\varepsilon \gg 1$

Substitute $-s^2/2\,\varphi''(c) = u^2$

$$= \sqrt{\frac{2}{-\varphi''(c)x}}\, f(c)e^{x\varphi(c)} \left(1+O\left(\frac{1}{\sqrt{x}}\right)\right)$$

$$\therefore \; I(x) \sim I_2(x) \sim \sqrt{\frac{2}{-\varphi''(c)x}}\, f(c)e^{x\varphi(c)} \quad \text{as } x \to \infty$$

4.6 Method of Stationary Phase

- Used when $\varphi = i\psi$, ψ real, so that

$$I(x) = \int_a^b f(t)\, e^{ix\psi(t)}\, dt.$$

Riemann-Lebesgue Lemma

If $\int_a^b |f(t)|\, dt < \infty$ and $\psi(t)$ is continuously differentiable for $t \in [a,b]$ and not constant on any sub-interval of $[a,b]$

then
$$\int_a^b f(t)\, e^{ix\psi(t)}\, dt \to 0 \quad \text{as } x \to \infty.$$

useful

- Usefl for integration by parts, eg.

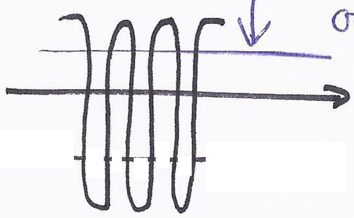
$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\, dt = \underline{-\frac{ie^{ix}}{2ix} + \frac{i}{x}} - \frac{i}{x} \underbrace{\int_0^1 \frac{e^{ixt}}{(1+t)^2}\, dt}_{\to 0 \text{ as } x\to\infty \text{ by RLL.}}$$

First term of an asymptotic expansion

- Why does RLL hold?

(i) For $\psi(t) = t$.

$$\int_a^b f(t)\, \underbrace{e^{ixt}}_{\substack{\text{oscillates}\\\text{more and}\\\text{more rapidly}}}\, dt$$

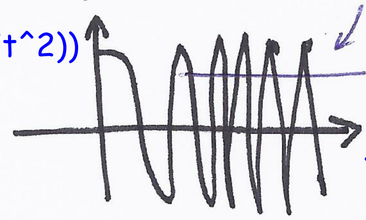


$f(t) \doteq$ const. over period $\therefore$ Cancels out.

(ii) More generally.

Near $t = t_0$, $\psi(t) \sim \psi(t_0) + (t - t_0)\psi'(t_0) + \cdots$

Period of oscillation $\sim \dfrac{2\pi}{x|\psi'(t_0)|}$



$\to 0$ as $x \to \infty$ providing $|\psi'(t_0)| \neq 0$

$\therefore$ Again get cancellation, unless $|\psi'(t_0)| = 0$

Unless ψ is constant on a region of non-zero measure, a stationary point is not enough to save the integral as $x \to \infty$, and one gets zero.

Nonetheless the dominant terms for x large but not infinite are from where $|\psi'(t_0)| = 0$

Example

$\psi''(t) \sim$ ord(1) in neighbourhood of c.

$f(c) \neq 0$; $\psi'(c) = 0$, $c \in (a,b)$; $\psi'(t) \neq 0$ $t \in [a,b] \backslash \{c\}$.

$$I(x) = \Bigg[\underbrace{\int_a^{c-\varepsilon}}_{I_1(x)} + \underbrace{\int_{c-\varepsilon}^{c+\varepsilon}}_{I_2(x)} + \underbrace{\int_{c+\varepsilon}^{b}}_{I_3(x)} \Bigg] f(t) e^{ix\psi(t)}\, dt$$

$\varepsilon \ll 1$

$$I_2(x) = \int_{c-\varepsilon}^{c+\varepsilon} dt \left[f(c) + O(t-c)\right] \exp\left[ix\left\{\psi(c) + \tfrac{1}{2}(t-c)^2\psi''(c) + O\left((t-c)^3\right)\right\}\right].$$

1) Isolate dominant contribution (no longer need be a maximum) and reduce range of integration to this region.

$$= e^{ix\psi(c)} \int_{c-\varepsilon}^{c+\varepsilon} dt \left[f(c)+O(t-c)\right] e^{\frac{ix}{2}(t-c)^2\psi''(c)} \left(1+O\left((t-c)^3 x\right)\right)$$

need to check errors in the approx ... harder here ... will do this at the end

providing $\varepsilon^3 x \ll 1$

$\therefore \varepsilon \ll 1/x^{1/3}$

2) Taylor expand and rescale

$x(t-c)^2 = s^2$

$$= \frac{e^{ix\psi(c)}}{\sqrt{x}} \int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} ds \left(f(c) + \underbrace{O\left(\tfrac{s}{\sqrt{x}}\right)}_{\text{subleading}}\right) e^{is^2\psi''(c)/2} \left(1+\underbrace{O\left(s^3/\sqrt{x}\right)}_{\text{subleading}}\right)$$

Drop

need to check scale of errors in the approximations ... will do this at the end

3) Extend range of integration

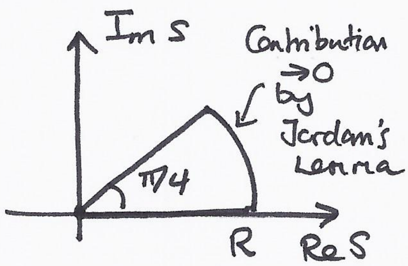
$$= \frac{e^{ix\psi(c)}}{\sqrt{x}} f(c) \int_{-\infty}^{\infty} ds\, e^{is^2\psi''(c)/2} + \dots$$

Requires $\varepsilon\sqrt{x} \gg 1$

$$\boxed{\frac{1}{x^{1/2}} \ll \varepsilon \ll 1/x^{1/3}}$$

$$\int_{-\infty}^{\infty} ds\, e^{is^2\psi''(c)/2} = 2\int_0^{\infty} ds\, e^{is^2\psi''(c)/2}$$

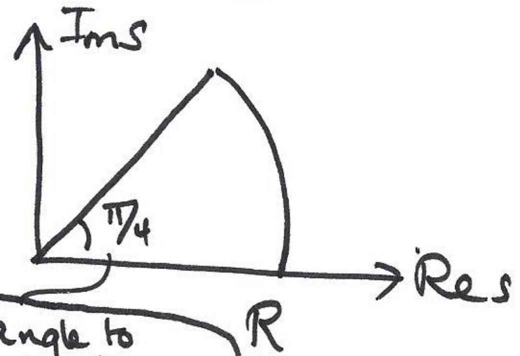
$$= \left(\frac{2\pi}{|\psi''(c)|}\right)^{1/2} e^{i\pi/4\, \mathrm{sgn}(\psi''(c))}$$



$\psi''(c) > 0$

Angle $-\pi/4$ for $\psi''(c) < 0$

$\boxed{\text{With } \psi''(c) > 0}$



take angle to be $-\pi/4$ if $\psi''(c)<0$

$$\overset{\text{Cauchy}}{0} = \oint_{R} ds\, e^{is^2\psi''(c)/2}$$

$$= \left[\int_{\rightarrow} + \int_{\swarrow}\right] ds\, e^{is^2\psi''(c)/2}$$

$s^2 = e^{i\pi/2}p$ $\quad s = e^{i\pi/4}p$

using $\int_{R} \to 0$ as $R\to\infty$ by Jordan's Lemma.

$$\therefore \int_0^\infty ds\, e^{is^2\psi''(c)/2} = \int_0^\infty dp\, e^{-p^2\psi''(c)/2} \cdot e^{i\pi/4}$$

$$= e^{i\pi/4}\sqrt{\frac{\pi}{2\psi''(c)}} \qquad \psi''(c)>0$$

More generally

$$\int_0^\infty ds\, e^{is^2\psi''(c)/2} = e^{i\pi/4\,\mathrm{sgn}(\psi''(c))}\sqrt{\frac{\pi}{2|\psi''(c)|}}$$

$$\therefore\ I_2(x) = \left(\frac{2\pi}{|\psi''(c)|}\right)^{1/2} \exp\left[i\pi/4\ \mathrm{sgn}(\psi''(c))\right] \frac{e^{ix\psi''(c)}}{\sqrt{x}} f(c) + \dots$$

$$\therefore \quad I_2(x) = \left(\frac{2\pi}{|\psi''(c)|}\right)^{1/2} \exp\left[i\pi/4 \; \mathrm{sgn}(\psi''(c))\right] \frac{e^{ix\psi''(c)}}{\sqrt{x}} f(c) + \dots$$

Size of Correction terms

1) Corrections from change of limits

$$\int_{\varepsilon\sqrt{x}}^{\infty} e^{is^2\psi''(c)/2}\,ds = \int_{\varepsilon\sqrt{x}}^{\infty} \frac{ds}{is\psi''(c)} \, is\psi''(c) e^{is^2\psi''(c)/2}$$

$$= \left[\frac{1}{is\psi''(c)} e^{is^2\psi''(c)/2}\right]_{\varepsilon\sqrt{x}}^{\infty} - \underbrace{\int_{\varepsilon\sqrt{x}}^{\infty} \frac{-1}{is^2\psi''(c)} e^{is^2\psi''(c)/2}\,ds}_{\text{Smaller correction}}$$

$$= O\left(\frac{1}{\varepsilon\sqrt{x}}\right) \qquad (\text{note } \varepsilon\sqrt{x} \gg 1).$$

Similar contribution from $\displaystyle\int_{-\infty}^{-\sqrt{x}\varepsilon} e^{is^2\psi''(c)/2}\,ds$

2) Corrections from Taylor Expansions

Expansion in $(t-c)^3 x = s^3/\sqrt{x}$

$$\underbrace{\frac{1}{\sqrt{x}}\int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} \frac{s^n}{x^{n/2}} e^{is^2\psi''(c)/2}\,ds}, \qquad \underbrace{\frac{1}{\sqrt{x}}\int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} \frac{(s^3)^n}{x^{n/2}} e^{is^2\psi''(c)/2}}$$

$\boxed{n \geq 1}$

$$\frac{1}{\sqrt{x}} \frac{1}{x^{n/2}} (\sqrt{x}\varepsilon)^{n-1} \sim \frac{\varepsilon^{n-1}}{x} \qquad \frac{1}{\sqrt{x}}\frac{1}{x^{n/2}} (\sqrt{x}\varepsilon)^{3n-1} \sim \frac{(\varepsilon^3 x)^n}{x\varepsilon} \sim \frac{\varepsilon(\varepsilon^3 x)^n}{x\varepsilon^2}$$

Using $\displaystyle\int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} s^n e^{is^2\psi''(c)/2}\,ds = O\left((\sqrt{x}\varepsilon)^{n-1}\right)$ by parts.

3) Correction from $I_1(x)$

$$I_1(x) = \int_a^{c-\varepsilon} f(t)e^{ix\psi(t)}\,dt \qquad \frac{1}{x^{1/2}} \ll \varepsilon \ll 1/x^{1/3}$$

$$= \int_a^{c-\varepsilon} \frac{f(t)}{ix\psi'(t)} \frac{\partial}{dt}\left(e^{ix\psi(t)}\right)dt$$

$$= \left[\frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)}\right]_a^{c-\varepsilon} - \frac{1}{ix}\underbrace{\underbrace{\int_a^{c-\varepsilon} e^{ix\psi(t)} \frac{\partial}{dt}\left(\frac{f(t)}{\psi'(t)}\right)dt}_{\to 0 \text{ as } x\to\infty \text{ by RLL if it exists;}}}_{o\left(\frac{1}{x}\right)}$$

$$\sim O\left(\frac{1}{x\psi'(c-\varepsilon)}\right)$$

$$\sim O\left(\frac{1}{\varepsilon x}\right)$$

$\psi'' \sim O(1)$ in neighbourhood of c.

Small "oh"

Similarly for I_3.

Note Corrections algebraically small, not exponentially small as in other methods

Next order terms very difficult to find?

in general need to consider whole integration domain not just behaviour near $t=c$

$$\boxed{\therefore\ I(x) \sim \left(\frac{2\pi}{|\psi''(c)|}\right)^{1/2} e^{i\pi/4\,\mathrm{sgn}(\psi''(c))} \frac{e^{ix\psi''(c)}}{\sqrt{x}} f(c)}$$

with corrections at $O\left(\frac{1}{\varepsilon\sqrt{x}}\right)$

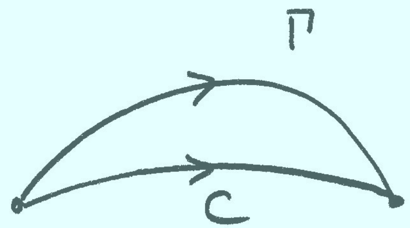
4.7 Method of Steepest Descents

- Generalises Laplace's method to consider

$$I(x) = \int_C f(t) e^{x\varphi(t)}\, dt \quad \text{as } x \to \infty, \ x \text{ real},$$

where $f(t), \varphi(t)$ are holomorphic (and thus analytic), with C a contour in the complex t plane.

- Key idea: $I(x)$ unchanged upon deforming C to a new contour Γ, with the same start and end points.



$$I(x) = \int_\Gamma f(t) e^{x\varphi(t)}\, dt$$

- If we find a contour Γ on which $\mathrm{Im}(\varphi(t))$ is piecewise constant, i.e. Γ_j, v_j such that $\Gamma = \bigcup_j \Gamma_j$ with $\mathrm{Im}\,\varphi(t) = v_j = \text{const}$ on Γ_j then

$$I(x) = \sum_j e^{ixv_j} \int_{\Gamma_j} f(t) e^{x \operatorname{Re} \varphi(t)} dt$$

and each integral can be analysed as $x \to \infty$ using Laplace's method.

- Let $\varphi(t) = u(\xi, \eta) + iv(\xi, \eta)$ with $t = \xi + i\eta$.
- As φ is holomorphic, we have the Cauchy Riemann Equations (CRE):

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta}, \quad \frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi}.$$

Hence $\nabla u \cdot \nabla v = u_\xi v_\xi + u_\eta v_\eta = 0$ $\therefore$ $\nabla u \perp \nabla v$

Also $\nabla v \perp$ contours with v const $\therefore$ Contours with v const $\parallel \nabla u$.

∇u points in direction u increases at fastest rate

$-\nabla u$ points in direction u decreases at fastest rate

$\therefore$ Contour with v constant is a path of steepest ascent/descent of u.

Landscape of $u(\xi,\eta)$

- CRE. $\quad u_{\xi\xi} + u_{\eta\eta} = (v_\eta)_\xi + (-v_\xi)_\eta = 0$

- Hence u cannot have a maximum or a minimum (unless we are also considering a point where u is singular or a branch point, where φ is not holomorphic).

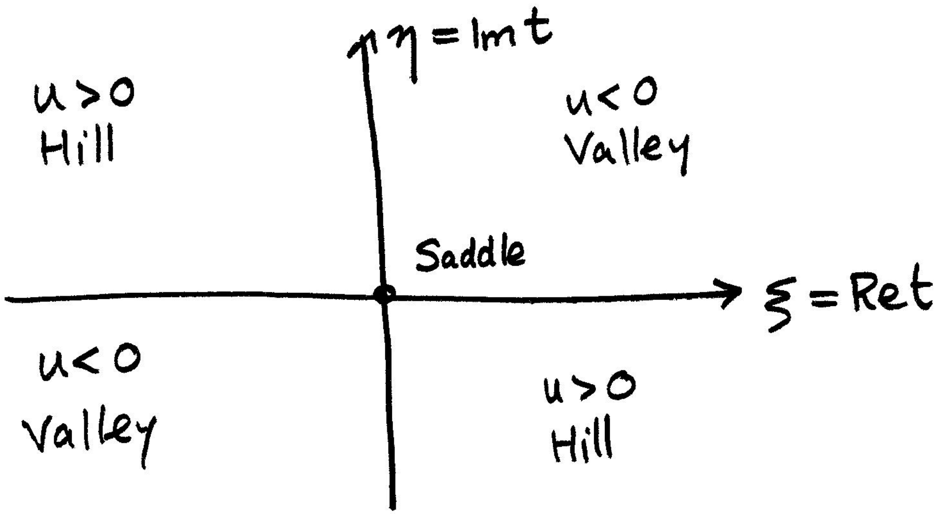
- At a stationary point, where $u_\xi = u_\eta = 0$, we have a SADDLE.

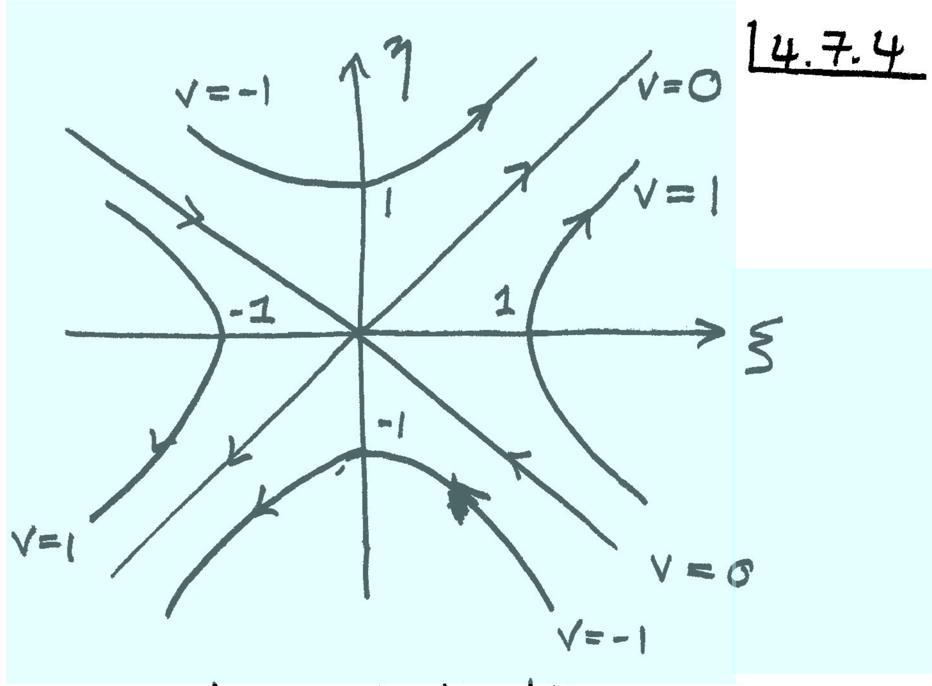
- Landscape of u has hills $(u>0)$, valleys $(u<0)$ at infinity with saddle points in the interior of the complex plane.

Example

$$\varphi(t) = it^2 = i(\xi+i\eta)^2 = -2\xi\eta + i(\xi^2-\eta^2) \quad \therefore u = -2\xi\eta\,,\ v = \xi^2-\eta^2$$

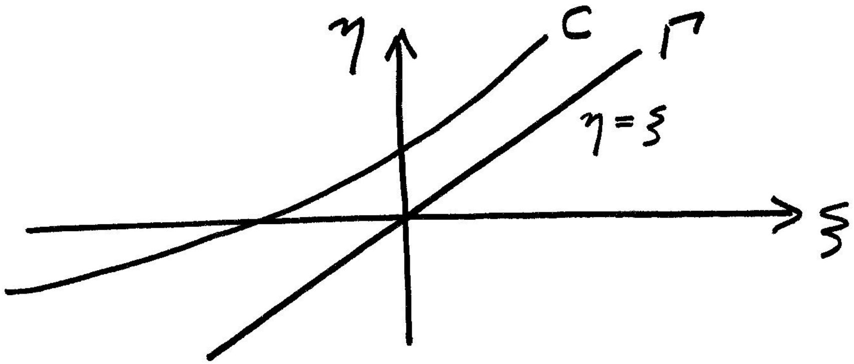
$$\nabla u = -2(\eta,\xi) \quad \therefore \text{ Saddle point at } \xi=\eta=0$$





Arrows in direction of decreasing u with STEEPEST DESCENT

- Contour C infinite, with endpoints in different valleys.
  - If endpoints not in valleys, integral $I(x)$ not well defined.



Deform C into Γ.

Integrals at infinity subleading

Hence method known as "Method of steepest descents" or saddle point method

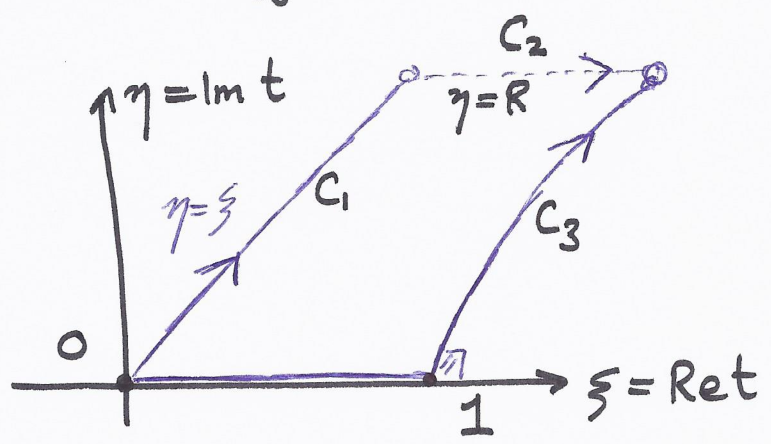
To use the method ...

* Deform contour to union of steepest descent (v const) contours through the endpoints and any relevant saddle points

* Evaluate local contributions from saddle and end points using Laplace's method.

Example

$$I(x) = \int_0^1 e^{x\varphi(t)}\,dt \quad \text{as } x\to\infty, \text{ with } \varphi(t) = it^2.$$



Steepest descent contour through $t=0$ is $\eta = \xi$

Steepest descent contour through $t=1$ is $\xi^2 - \eta^2 = 1$

$$C_1(R) = \{\xi(1+i),\ \xi\in[0,R]\}$$

$$C_2(R) = \{\xi + iR,\ \xi\in[R, \sqrt{R^2+1}]\}$$

$$C_3(R) = \{\sqrt{1+\eta^2} + i\eta,\ \eta\in[0,R]\}$$

$$\therefore\ I(x) = \left[\int_{C_1(R)} + \int_{C_2(R)} - \int_{C_3(R)}\right] e^{ixt^2}\,dt$$

On $C_2(R)$
$$|\exp(ixt^2)| = |\exp(ix(\xi^2 - R^2 + 2i\xi R))|$$
$$= |\exp(-2x\xi R)| = O(e^{-2xR^2}) \to 0$$
as $R\to\infty$.

$$\therefore\ \int_{C_2(R)} e^{ixt^2}\,dt \to 0 \quad \text{as } R\to\infty.$$

$$\int_{C_1(\infty)} e^{ixt^2}dt = \int_0^\infty \exp\left(ix\xi^2(1+i)^2\right)d\xi\,(1+i)$$

$i(1+i)^2 = i(1+2i+i^2)$
$= 2i^2 = -2.$

$$= (1+i)\int_0^\infty e^{-2x\xi^2}d\xi \qquad u = \sqrt{2x}\,\xi$$

$$= \frac{1+i}{\sqrt{2}\sqrt{x}}\int_0^\infty e^{-u^2}du = \frac{e^{i\pi/4}}{2}\sqrt{\frac{\pi}{x}}.$$

$$\int_{C_3(\infty)} e^{ixt^2}dt = \int_0^\infty e^{ix\underbrace{\left[(1+\eta^2)^{1/2}+i\eta\right]^2}_{1+2i\eta(1+\eta^2)^{1/2}}}\frac{dt}{d\eta}\,d\eta$$

$$= e^{ix}\int_0^\infty e^{x\varphi(\eta)}f(\eta)\,d\eta$$

with $\varphi(\eta) = -2\eta(1+\eta^2)^{1/2}$,

$$f(\eta) = \frac{dt}{d\eta} = \frac{\eta}{(1+\eta^2)^{1/2}} + i$$

and thus Laplace's method can be used.

However, we can get to a quicker answer, at all orders, by noting

on $C_3(\infty)$, $\quad t = \xi + i\eta \quad$ where $\xi^2 - \eta^2 = 1$

$$\therefore\ t^2 = \xi^2 - \eta^2 + 2i\xi\eta = 1 + 2i\eta(1+\eta^2)^{1/2}$$

$\therefore$ Let $t^2 = 1 + is \qquad s \in [0,\infty)$

$\therefore$ $\underline{\underline{t = (1+is)^{1/2}}}$ (principal branch ie +ve squareroot).

$\boxed{4.7.8}$

$dt/ds = \frac{1}{2} i \frac{1}{(1+is)^{1/2}}$

Then

$$\int_{C_3(\infty)} e^{ixt^2}\,dt = \int_0^\infty e^{ix} \cdot e^{-xs} \frac{dt}{ds}\,ds$$

$$= \frac{ie^{ix}}{2} \int_0^\infty e^{-xs} \frac{1}{(1+is)^{1/2}}\,ds$$

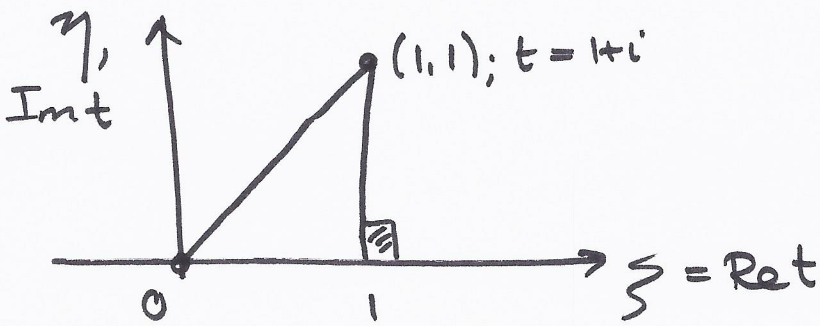
$$\underset{\text{Lemma}}{\overset{\text{Watson's}}{\sim}} \frac{ie^{ix}}{2} \sum_{n=0}^{\infty} \frac{a_n \Gamma(n+1)}{x^{n+1}} \quad \text{as } x \to \infty$$

$$\text{with } a_n = \frac{(-i)^n \Gamma(n+\frac{1}{2})}{\Gamma(n+1)\sqrt{\pi}}$$

$$\therefore I(x) \sim \frac{e^{i\pi/4}}{2}\sqrt{\pi/x} - \frac{ie^{ix}}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-i)^n \Gamma(n+\frac{1}{2})}{x^{n+1}} \quad \text{as } x \to \infty$$

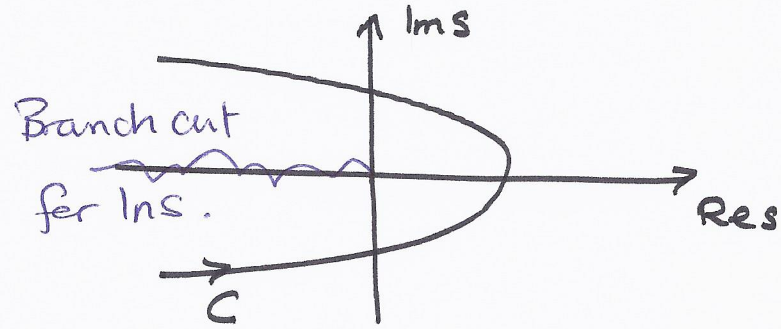
<u>Note</u>

Local contributions dominate ... just need to get tangents to steepest descent paths ... eg. could use



in the above example.

Example



$$I(x) = \int_C e^s s^{-x}\, ds$$

as $x \to \infty$

Note $\quad e^s s^{-x} = \exp\left[s - x\log s\right]$, branch cut for $\log s$, is given by $\{\mathrm{Re}\, s<0, \mathrm{Im}\, s=0\}$

Let $s = tx$ $\quad$ Saddle point at $x/s = 1$. Fix saddle point location by setting $t = s/x$.

$$I(x) = x\int_{C_*} dt\, \underbrace{e^{tx - x\log(tx)}}_{e^{tx - x\log t - x\log x}} = x^{1-x}\int_{C_*} dt\, e^{x\varphi(t)}$$

with $\varphi(t) = t - \log t$.

$\therefore \varphi = \xi + i\eta - \log r - i\theta$

polars.

$0 = \varphi'(t) = 1 - 1/t \quad \therefore$ Saddle at $t = 1$

Deform C_* through this point

$$u = \mathrm{Re}\,\varphi = r\cos\theta - \log r \qquad v = \mathrm{Im}\,\varphi = r\sin\theta - \theta$$

At $t=1$ $\quad \theta = 0,\ v = 0$

$\therefore$ Path of steepest descent through $t=1$ given by

$$r = \frac{\theta}{\sin\theta} \qquad \theta \in (-\pi, \pi)$$

On this path, Γ $\quad u = \mathrm{Re}\,\varphi = r(\theta)\cos\theta - \log r(\theta)$

$$= \theta\cot\theta - \log\theta + \log\sin\theta$$

4.7.10

$$\therefore\ I(x) = x^{1-x} \int_{-\pi}^{\pi} e^{x u(\theta)} \frac{dt}{d\theta}\, d\theta$$

$t = r(\theta)e^{i\theta}$

$dt/d\theta = (r'(\theta) + i r(\theta))\, e^{i\theta}$

$$= x^{1-x} \int_{-\pi}^{\pi} d\theta\, e^{x\{\underbrace{\theta\cot\theta - \log\left(\frac{\theta}{\sin\theta}\right)}_{\Psi(\theta)}\}} \underbrace{\left[r'(\theta) + i r(\theta)\right] e^{i\theta}}_{F(\theta)}$$

Laplace's method with interior maximum at $\theta = 0$

$$I(x) \sim x^{1-x} \frac{\sqrt{2\pi}\, F(0) e^{x\Psi(0)}}{\sqrt{-\Psi''(0)x}} \quad \text{as } x \to \infty$$

By Taylor expanding, $r(\theta) = \dfrac{\theta}{\sin\theta} = \dfrac{\theta}{\theta - \theta^3/3! + \cdots} = 1+\theta^2/6+O(\theta^3)$

and hence $F(0) = i$

$$\Psi(\theta) = \frac{\theta(1-\theta^2/2! + \cdots)}{\theta - \theta^3/3! + \cdots} - \log\left(1+\theta^2/6+O(\theta^3)\right)$$

$$= 1 - \theta^2/2 + O(\theta^3)$$

$$\Psi(0) = 1 \qquad \Psi''(0) = -1$$

$$\therefore\ I(x) \sim i x^{1/2 - x} e^{x} \sqrt{2\pi} \quad \text{as } x \to \infty$$

NB this example can be used to deduce $\Gamma(x) \sim \sqrt{2\pi}\, x^{x-1/2} e^{-x}$, ie. Stirling's approx... see online notes.

4.8 Splitting Integration Range

* Previously, have split integration range to isolate dominant contribution
* More generally, can split integration range and use different approximations in each range

$x \sim O(1)$ Integrand $O(1)$
Integration range $O(1)$
Integral $O(1)$

Do not lecture this example.

Example $$I(\varepsilon) = \int_0^1 \frac{dx}{(x+\varepsilon)^{1/2}} \quad \text{as } \varepsilon \to 0^+.$$

$x \sim O(\varepsilon)$ Integrand $O(1/\varepsilon^{1/2})$

$x = \mathrm{ord}(1)$
$$\frac{1}{(x+\varepsilon)^{1/2}} = \frac{1}{x^{1/2}} \frac{1}{(1+\varepsilon/x)^{1/2}}$$
$$= \frac{1}{x^{1/2}}\left(1 - \frac{\varepsilon}{2x} + O\left(\varepsilon^2/x^2\right)\right)$$
as $\varepsilon \to 0$

Expansion not valid for $x \sim O(\varepsilon)$

$\therefore$ Split.
$$I(x) = \underbrace{\int_0^\delta \frac{dx}{(x+\varepsilon)^{1/2}}}_{I_1} + \underbrace{\int_\delta^1 \frac{dx}{(x+\varepsilon)^{1/2}}}_{I_2} \qquad \varepsilon \ll \delta \ll 1$$

$O(1)$

$$I_2 = \int_\delta^1 dx \left(\frac{1}{x^{1/2}} - \frac{\varepsilon}{2x^{3/2}} + O\left(\varepsilon^2/x^{5/2}\right)\right)$$
okay as $\frac{\varepsilon}{x} < \varepsilon/\delta \ll 1$

$$= 2(1-\delta^{1/2}) + \varepsilon\left(1 - \frac{1}{\sqrt{\delta}}\right) + O\left(\varepsilon^2/\delta^{3/2}\right)$$

$$I_1 = \int_0^\delta \frac{dx}{(x+\varepsilon)^{1/2}} \qquad \text{Let } x = \varepsilon u$$

$$= \int_0^{\delta/\varepsilon} \frac{\varepsilon\, du}{\varepsilon^{1/2}(1+u)^{1/2}} = 2\varepsilon^{1/2}\left(1+\delta/\varepsilon\right)^{1/2} - 2\varepsilon^{1/2}$$

$\varepsilon/\delta \ll 1$

$$= 2\delta^{1/2}\left(1+\varepsilon/\delta\right)^{1/2} - 2\varepsilon^{1/2}$$

$$= 2\delta^{1/2} + \varepsilon/\delta^{1/2} + O\left(\varepsilon^2/\delta^{3/2}\right) - 2\varepsilon^{1/2}$$

$$\therefore I = I_1 + I_2 = 2 - 2\delta^{1/2} + \varepsilon - \varepsilon/\delta^{1/2} + O\left(\varepsilon^2/\delta^{3/2}\right)$$
$$+ 2\delta^{1/2} \qquad + \varepsilon/\delta^{1/2} \quad - 2\varepsilon^{1/2} + O\left(\varepsilon^2/\delta^{3/2}\right)$$

$$= 2 - 2\varepsilon^{1/2} + \varepsilon + \cdots \qquad \text{noting } \varepsilon \ll \delta$$

$$\therefore \frac{\varepsilon^2}{\delta^{3/2}} = \frac{\varepsilon^2}{\delta^2}\delta^{1/2} \ll 1$$

NB Exact value

$$I(\varepsilon) = 2\left((1+\varepsilon)^{1/2} - \varepsilon^{1/2}\right) = 2 - 2\varepsilon^{1/2} + \varepsilon + \cdots$$

Example

$$I(\varepsilon) = \int_0^{\pi/4} \frac{d\theta}{\varepsilon^2 + \sin^2\theta} \qquad \text{as } \varepsilon \to 0^+$$

$\theta \sim O(1)$ Integrand $\sim O(1)$ Integral $\sim O(1)$

$\theta \sim O(\varepsilon)$ Integrand $\sim O(\frac{1}{\varepsilon^2})$ Integration range $\sim O(\varepsilon)$

Integral $\sim O(1/\varepsilon)$

<u>Split</u>

$$I = \underbrace{\int_0^\delta \frac{d\theta}{\varepsilon^2 + \sin^2\theta}}_{I_1} + \underbrace{\int_\delta^{\pi/4} \frac{d\theta}{\varepsilon^2+\sin^2\theta}}_{I_2}$$

$\varepsilon \ll \delta \ll 1$

need this to be small ... thus need

$$I_2 = \int_\delta^{\pi/4} \left(\frac{1}{\sin^2\theta} + O\left(\frac{\varepsilon^2}{\sin^4\theta}\right)\right) d\theta$$

$\varepsilon^2/\delta^4 \ll 1$

i.e. $\varepsilon^{1/2} \ll \delta \ll 1$

$$= -\left[\cot\theta\right]_\delta^{\pi/4} + O(\varepsilon^2/\delta^3)$$

$$= -1 + \frac{(1-\delta^2/2+\dots)}{\delta - \delta^3/6 + \dots} + O(\varepsilon^2/\delta^3) = -1 + \frac{1}{\delta} + O(\delta) + \underbrace{O(\varepsilon^2/\delta^3)}$$

$\ll 1$ for $\varepsilon^{2/3} \ll \delta \ll 1$

consistent with $\varepsilon^{1/2} \ll \delta \ll 1$

wlog $\varepsilon > 0$ $\quad \theta = \varepsilon u$

$\varepsilon u \leq \varepsilon \cdot \frac{\delta}{\varepsilon} = \delta \ll 1$

$$I_1 = \int_0^{\delta/\varepsilon} \frac{\varepsilon\, du}{\varepsilon^2 + \sin^2(\varepsilon u)}$$

$$= \varepsilon \int_0^{\delta/\varepsilon} \frac{du}{\varepsilon^2 + \varepsilon^2 u^2 + O(\varepsilon^4 u^4)}$$

$$I_1 = 1/\varepsilon \int_0^{\delta/\varepsilon} du \left[\frac{1}{1+u^2} + O\left(\frac{\varepsilon^2 u^4}{(1+u^2)^2} \right) \right]$$

$$= \frac{1}{\varepsilon} \tan^{-1}(\delta/\varepsilon) + O\left(\frac{1}{\varepsilon} \cdot \frac{\delta}{\varepsilon} \cdot \varepsilon^2 \right)$$

$$= \frac{1}{\varepsilon}\left[\pi/2 - \frac{1}{\delta/\varepsilon} + O\left(\frac{1}{(\delta/\varepsilon)^2} \right) \right] + O(\delta)$$

$$= \frac{\pi}{2\varepsilon} - \frac{1}{\delta} + \underbrace{O\left(\varepsilon/\delta^2 \right)}_{\ll 1 \text{ for } \varepsilon^{1/2} \ll \delta \ll 1} + O(\delta)$$

$$\therefore \quad I = I_1 + I_2 = -1 + 1/\delta + \frac{\pi}{2\varepsilon} - \frac{1}{\delta} + \underbrace{O\left(\frac{\varepsilon^2}{\delta^3}, \frac{\varepsilon}{\delta^2}, \delta \right)}_{\ll 1}$$

$$= \frac{\pi}{2\varepsilon} - 1 + \cdots \quad \text{as } \varepsilon \to 0$$

5. Matched Asymptotic Expansions

5.1 Singular Perturbations

Example $\qquad \varepsilon y'' + y' + y = 0 \qquad 0 < x < 1, \quad y(0) = a, \; y(1) = b.$

$\varepsilon = 0$ $\qquad y' + y = 0$. Hence $y = Ae^{-x}$, which cannot satisfy both boundary conditions in general.

This is a singular perturbation problem.

More generally suppose D_ε is a differential operator that depends on a small parameter ε, e.g. $D_\varepsilon = \varepsilon \, d^2/dx^2 + d/dx + 1$.

Then a differential equation $D_\varepsilon y = 0$ with boundary conditions is a singular perturbation problem if the order of $D_0 y$ is less than the order of $D_\varepsilon y$ as $\varepsilon \to 0$

[Since the solution of $D_0 y$ cannot satisfy BCs in general].

Suppose $D_\varepsilon = \varepsilon \frac{d^k}{dx^k}$ + lower order derivatives.

* Over most of the range, $\varepsilon \frac{d^k y}{dx^k}$ is small and y satisfies $D_0 y = 0$ to good approximation.

* In some regions, typically near boundaries, $\varepsilon \frac{d^k y}{dx^k}$ is not small and y adjusts to satisfy BCs.

The usual procedure for finding a solution to a singular ODE problem is:

(*) Determine the scaling in the boundary layers e.g.

$$x = \varepsilon \hat{x} \qquad \text{or} \qquad x = \varepsilon^{1/2} \hat{x}$$

(*) Find the asymptotic expansions in the boundary layers ("inner" solution) and outside the boundary layers ("outer" solutions).

(*) Fix the constants of integration in these solutions by

- demanding the inner solutions satisfy the BCs
- "matching" - ensuring the expansion of the inner and outer solutions agree in an overlap region between them.

This is the method of Matched Asymptotic Expansions

Previous Example $\qquad \varepsilon y'' + y' + y = 0 \qquad 0 < x < 1, \quad y(0) = a, \; y(1) = b.$

Left Hand Boundary Scaling

Let $x = \varepsilon^{\alpha} x_L$ $\quad y(x) = y_L(x_L)$ $\quad$ with $\alpha > 0$.

$\therefore$ $dy/dx = \dfrac{1}{\varepsilon^{\alpha}} \dfrac{dy_L}{dx_L}$ and $\quad \varepsilon^{1-2\alpha} \dfrac{d^2 y_L}{dx_L} + \varepsilon^{-\alpha} \dfrac{dy_L}{dx_L} + y_L = 0$

Dominant balance $\quad 1 - 2\alpha = -\alpha \quad \therefore \alpha = 1$. Hence boundary layer has width of ord(ε).

Right Hand Boundary Layer: Proceeds similary with $x = 1 + \varepsilon^{\beta} x_R$, $y(x) = y_R(x_R)$.
one finds $\beta = 1$.

Develop asymptotic solution

(1) Away from boundary layers (outer region), expand $y(x) \sim y_{out,0}(x) + \varepsilon y_{out,1}(x) + \cdots$
as $\varepsilon \to 0^+$ with $x, 1-x = \text{ord}(1)$

(2) Left Hand Boundary. Let $x = \varepsilon x_L$ and expand

$$y(x) = y_L(x_L) \sim y_{L,0}(x_L) + \varepsilon y_{L,1}(x_L) + \cdots \quad \text{as } \varepsilon \to 0^+ \text{ with } x_L = \text{ord}(1).$$

(3) Right hand boundary. Let $x = 1+\varepsilon x_R$ and expand

$$y(x) = y_R(x_R) \sim y_{R,0}(x_R) + \varepsilon y_{R,1}(x_R) + \cdots \quad \text{as } \varepsilon \to 0^+ \text{ with } -x_R \sim \text{ord}(1)$$

Left hand boundary layer

$$\frac{d^2 y_L}{dx_L^2} + \frac{dy_L}{dx_L} + \varepsilon y_L = 0, \quad x_L > 0.$$

$O(\varepsilon^0)$ $$\frac{d^2 y_{L,0}}{dx_L^2} + \frac{dy_{L,0}}{dx_L} = 0, \quad x_L > 0.$$

$O(\varepsilon^1)$ $$\frac{d^2 y_{L,1}}{dx_L^2} + \frac{dy_{L,1}}{dx_L} + y_{L,0} = 0, \quad x_L > 0.$$

$$\therefore \quad y_{L,0} = A_{L,0} + B_{L,0} e^{-x_L}$$

$$y_{L,1} = A_{L,1} + B_{L,1} e^{-x_L} + \left(B_{L,0} x_L e^{-x_L} - A_{L,0} x_L\right)$$

BC $y_{L,0}(0) = a, \quad y_{L,1}(0) = 0 \quad \therefore \quad A_{L,0} + B_{L,0} = a, \quad A_{L,1} + B_{L,1} = 0.$

Right hand boundary layer

$$\frac{d^2 y_R}{dx_R^2} + \frac{dy_R}{dx_R} + \varepsilon y_R = 0 \qquad x_R < 0$$

As with left hand layer
$$y_{R,0}(x_R) = A_{R,0} + B_{R,0} e^{-x_R} \qquad (x_R < 0)$$
$$y_{R,1}(x_R) = A_{R,1} + B_{R,1} e^{-x_R} + (B_{R,0} x_R e^{-x_R} - A_{R,0} x_R)$$

with $A_{R,0} + B_{R,0} = b$, $A_{R,1} + B_{R,1} = 0$

Outer region

$$\varepsilon \frac{d^2 y_{out}}{dx^2} + \frac{dy_{out}}{dx} + y_{out} = 0 \qquad 0 < x < 1$$

$O(\varepsilon^0)$ $$\frac{dy_{out,0}}{dx} + y_{out,0} = 0$$

$O(\varepsilon^1)$ $$\frac{dy_{out,1}}{dx} + y_{out,1} = -\frac{d^2 y_{out,0}}{dx^2}$$

Solve

$$y_{out,0} = A_{out,0} e^{-x}$$

$$y_{out,1} = A_{out,1} e^{-x} - A_{out,0} x e^{-x}$$

Instead of applying BCs at $x = 0, 1$, we need to match with the left and right boundary layer solutions

<u>Idea:</u> There is an overlap, or intermediate, region where both expansions hold and therefore are equal.

<u>Hence</u> Introduce an intermediate scaling, $x = \varepsilon^{\gamma}\hat{x}$ with $0<\gamma<1$. Then with $\hat{x}>0$, $\hat{x} = \text{ord}(1)$

$$x = \varepsilon^{\gamma}\hat{x} \to 0, \qquad x_L = \varepsilon^{\gamma-1}\hat{x} \to \infty \qquad \text{as } \varepsilon \to 0^+.$$

Matching requires expansions to be equal as $\varepsilon \to 0^+$ with $\hat{x}>0$, $\hat{x} = \text{ord}(1)$

i.e. $\quad y_L(\varepsilon^{\gamma-1}\hat{x}) \sim y_{out}(\varepsilon^{\gamma}\hat{x})$ as $\varepsilon \to 0^+$ with $\hat{x}>0$, $\hat{x} = \text{ord}(1)$

<u>We have</u>

$$y_L(\varepsilon^{\gamma-1}\hat{x}) = A_{L,0} + \underbrace{B_{L,0}\, e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\text{exponentially small}} + O(\varepsilon)$$

$$y_{out}(\varepsilon^{\gamma}\hat{x}) = A_{out,0}\, e^{-\varepsilon^{\gamma}\hat{x}} + O(\varepsilon) = A_{out,0}\left(1 - \varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma})\right) + O(\varepsilon)$$

<u>Same expansions</u> $\quad A_{L,0} = A_{out,0} \quad$ i.e. $\quad y_{L,0}(\infty) = y_{out,0}(0)$

<u>Matching outer and right hand boundary layer</u>

Let $x = 1 + \varepsilon^{\gamma}\hat{x}$ with $0 < \gamma < 1$. As $\varepsilon \to 0^+$, with $\hat{x} < 0$ and $\hat{x} = \mathrm{ord}(1)$

$$y_R(x_R = \varepsilon^{\gamma-1}\hat{x}) = A_{R,0} + \underbrace{B_{R,0}\, e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\substack{\text{exponential blow} \\ \text{up as } \varepsilon \to 0^+}} + O(\varepsilon)$$

$$\begin{aligned} y_{out}(x = 1 + \varepsilon^{\gamma}\hat{x}) &= A_{out,0}\, e^{-(1+\varepsilon^{\gamma}\hat{x})} + O(\varepsilon) \\ &= \frac{A_{out,0}}{e}\left(1 - \varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma})\right) + O(\varepsilon) \end{aligned}$$

Same expansions : $B_{R,0} = 0, \quad A_{out,0} = e A_{R,0}$

$$\therefore \left\{ \begin{matrix} A_{L,0} + B_{L,0} = a; & A_{R,0} + B_{R,0} = b \\ A_{L,0} = A_{out,0}; & B_{R,0} = 0; \; A_{out,0} = e A_{R,0} \end{matrix} \right\} \therefore \left\{ \begin{matrix} A_{L,0} = eb & ; \; A_{out,0} = eb \\ B_{L,0} = a - eb; & A_{R,0} = b; \; B_{R,0} = 0 \end{matrix} \right\}$$

$$\therefore \quad y_{L,0}(x_L) = eb + (a - eb)e^{-x_L}; \quad y_{out,0}(x) = ebe^{-x}; \quad y_{R,0}(x_R) = b.$$

Agreement with exact solution

Exact solution is $y(x) = A_+ e^{\lambda_- x} - A_- e^{\lambda_+ x}$ for $0 \le x \le 1$

with $A_\pm = \dfrac{a e^{\lambda_\pm} - b}{e^{\lambda_+} - e^{\lambda_-}}$, $\lambda_\pm = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$

Using expansions $\lambda_+ = -1 + O(\varepsilon)$; $\lambda_- = -\frac{1}{\varepsilon} + 1 + O(\varepsilon)$ as $\varepsilon \to 0^+$

can show

$$y(\varepsilon x_L) = y_{L,0}(x_L) + O(\varepsilon) \quad \text{as } \varepsilon \to 0^+ \text{ with } x_L > 0,\ x_L = \text{ord}(1)$$

$$y(x) = y_{out,0}(x) + O(\varepsilon) \quad \text{as } \varepsilon \to 0^+ \text{ with } 0 < x < 1 \text{ with } x,\ 1-x = \text{ord}(1)$$

$$y(1+\varepsilon x_R) = y_{R,0}(x_R) + O(\varepsilon) \quad \text{as } \varepsilon \to 0^+ \text{ with } x_R < 0\ \ x_R = \text{ord}(1).$$

Higher order Matching

Using the leading order solution, the first higher order solution is given by

$$y_{L,1}(x_L) = -eb\,x_L + (a-eb)\,x_L e^{-x_L} + A_{L,1} + B_{L,1} e^{-x_L}$$

$$y_{R,1}(x_R) = -b\,x_R + A_{R,1} + B_{R,1} e^{-x_R}$$

$$y_{out,1}(x) = -eb\,x e^{-x} + A_{out,1} e^{-x}$$

Recall BCs $\quad y_{L,1}(0)=0 \quad y_{R,1}(0)=0 \quad \therefore A_{L,1}+B_{L,1}=A_{R,1}+B_{R,1}=0$

Matching left hand boundary layer and outer region

As $\varepsilon\to 0^+$ with $\hat{x}>0$ $\quad \hat{x}=\mathrm{ord}(1)$ where $x=\varepsilon^{\gamma}\hat{x}$, $\quad 0<\gamma<1$

$$y_L(x_L=\varepsilon^{\gamma-1}\hat{x}) = y_{L,0}(\varepsilon^{\gamma-1}\hat{x}) + \varepsilon y_{L,1}(\varepsilon^{\gamma-1}\hat{x}) + o(\varepsilon^2)$$

$$= \left(eb+(a-eb)e^{-\varepsilon^{\gamma-1}\hat{x}}\right) + \varepsilon\left(-ebe^{\gamma-1}\hat{x}+(a-eb)\varepsilon^{\gamma-1}\hat{x}e^{-\varepsilon^{\gamma-1}\hat{x}} + A_{L,1}+B_{L,1}e^{-\varepsilon^{\gamma-1}\hat{x}}\right) + o(\varepsilon^2)$$

$$= eb - eb\varepsilon^{\gamma}\hat{x} + \varepsilon A_{L,1} + o(\varepsilon^2)$$

$$y_{out}(x=\varepsilon^{\gamma}\hat{x}) = y_{out,0}(\varepsilon^{\gamma}\hat{x}) + \varepsilon y_{out,1}(\varepsilon^{\gamma}\hat{x}) + o(\varepsilon^2)$$

$$= eb\underbrace{e^{-\varepsilon^{\gamma}\hat{x}}}_{(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma}))} + \varepsilon\left(-eb\varepsilon^{\gamma}\hat{x}\left(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma})\right) + A_{out,1}\left(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma})\right)\right) + O(\varepsilon^2)$$

$$\therefore\ y_{out}(x = \varepsilon^{\gamma}\hat{x}) = eb - eb\varepsilon^{\gamma}\hat{x} + \varepsilon A_{out,1} + O(\varepsilon^{1+\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2})$$

need $\gamma > 1/2$ to ensure $\varepsilon^{2\gamma}$ term subleading compared to $O(\varepsilon)$ term

Same expansions $\quad A_{L,1} = A_{out,1}$

Note some terms jump order eg. $-eb\varepsilon^{\gamma}\hat{x}$ arises from $y_{out,0}$ even though it's higher order and arises from $y_{L,1}$ in the expansion of the left inner

Matching Right hand boundary layer and outer

- As $\varepsilon \to 0^+$ with $\hat{x} < 0$, $\hat{x} = \text{ord}(1)$, $x_R = \varepsilon^{\gamma-1}\hat{x}$

$$\begin{aligned}
y_R(x_R = \varepsilon^{\gamma-1}\hat{x}) &= y_{R,0}(\varepsilon^{\gamma-1}\hat{x}) + \varepsilon y_{R,1}(\varepsilon^{\gamma-1}\hat{x}) + O(\varepsilon^2) \\
&= b + \varepsilon\left(-b\varepsilon^{\gamma-1}\hat{x} + A_{R,1} + B_{R,1}e^{-\varepsilon^{\gamma-1}\hat{x}}\right) + O(\varepsilon^2) \\
&= \underbrace{\varepsilon B_{R,1}e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\text{Exponentially leading term}} + b - \varepsilon^{\gamma}b\hat{x} + \varepsilon A_{R,1} + O(\varepsilon^2)
\end{aligned}$$

As $\varepsilon \to 0$ with $\hat{x} < 0$, $\hat{x} = \text{ord}(1)$, $x = 1 + \varepsilon^\gamma \hat{x}$

$$y_{out}(x = 1 + \varepsilon^\gamma \hat{x}) = y_{out,0}(1 + \varepsilon^\gamma \hat{x}) + \varepsilon y_{out,1}(1 + \varepsilon^\gamma \hat{x}) + O(\varepsilon^2)$$

$$= ebe^{-(1+\varepsilon^\gamma \hat{x})} + \varepsilon\left(-eb(1+\varepsilon^\gamma \hat{x})e^{-(1+\varepsilon^\gamma \hat{x})} + A_{out,1} e^{-(1+\varepsilon^\gamma \hat{x})}\right) + O(\varepsilon^2)$$

$$= b\left(1 - \varepsilon^\gamma \hat{x} + O(\varepsilon^{2\gamma})\right) + \varepsilon\left(-b(1+\varepsilon^\gamma \hat{x})\left(1 - \varepsilon^\gamma \hat{x} + O(\varepsilon^{2\gamma})\right) + \frac{A_{out,1}}{e}\left(1 - \varepsilon^\gamma \hat{x} + O(\varepsilon^{2\gamma})\right)\right) + O(\varepsilon^2)$$

$$= b - \varepsilon^\gamma b \hat{x} - \varepsilon b + \varepsilon A_{out,1}/e + O(\varepsilon^{2\gamma}, \varepsilon^{1+\gamma}, \varepsilon^2)$$

As before, $\gamma > 1/2$.

Same expansions: $\boxed{A_{R,1} = A_{out,1}/e - b \; ; \; B_{R,1} = 0}$

Hence
$$\left\{\begin{array}{ll} \underline{\text{BCs}} & A_{L,1} + B_{L,1} = A_{R,1} + B_{R,1} = 0 \\ \underline{\text{Matching}} & A_{L,1} = A_{out,1} \; ; \; B_{R,1} = 0 \; ; \; A_{R,1} = A_{out,1}/e - b \end{array}\right\} \therefore \left\{\begin{array}{l} A_{R,1} = B_{R,1} = 0 \\ A_{out,1} = A_{L,1} = -B_{L,1} \\ \qquad = eb \end{array}\right\}$$

Thus
$$y_{L,1}(x_L) = -ebx_L + (a-eb)x_L e^{-x_L} + eb(1-e^{-x_L})$$
$$y_{out,1}(x) = -ebxe^{-x} + ebe^{-x}$$
$$y_{R,1}(x) = -bx_R.$$

Note
$$\lim_{x\to 1} y_{out}(x) = \lim_{x\to 1}\left(ebe^{-x} + \varepsilon eb(1-x)e^{-x} + O(\varepsilon^2)\right) = b + O(\varepsilon^2)$$
$$\lim_{x\to 0} y_{out}(x) = \lim_{x\to 0}\left(ebe^{-x} + \varepsilon eb(1-x)e^{-x} + O(\varepsilon^2)\right) = eb + O(\varepsilon)$$

$\therefore$ $y_{out}(x)$ satisfies BC at $x=1$, at least to $O(\varepsilon^2)$ $\therefore$ Boundary layer not required at $x=1$.

However $\lim_{x\to 0} y_{out}(x) \neq a$ $\therefore$ Boundary layer at $x=0$ required.

Van Dyke's Matching Rule

- Using the intermediate variable $\hat{x}$ yields long calculations
- Van Dyke's matching rule is quicker and usually works :

$$\underbrace{m \text{ terms inner}\left[(n \text{ terms outer})\right]} = \underbrace{n \text{ terms outer}\left[(m \text{ terms inner})\right]}$$

5.14

n terms in the outer expansion, written in terms of the inner variable and expanded to m^{th} order in the inner variable = m terms in the inner expansion written in terms of the outer variable and expanded to n^{th} order in the outer variable

Example At the left hand boundary. $y_L(x_L) = A_{L,0} + (a - A_{L,0})e^{-x_L} + O(\varepsilon)$

$y_{out}(x) = A_{out,0}\, e^{-x} + O(\varepsilon)$, $\quad x = \varepsilon x_L$

LHS 1 term outer $= A_{out,0}\, e^{-x}$

$= A_{out,0}\, e^{-\varepsilon x_L}$

$= A_{out,0}\,(1 + O(\varepsilon x_L))$

RHS 1 term inner $= A_{L,0} + (a - A_{L,0})e^{-x_L}$

$= A_{L,0} + (a - A_{L,0})e^{-x/\varepsilon} = A_{L,0}$ + exponentially small

$\therefore$ $A_{out,0}$ = 1 term inner [(1 term outer)] = 1 term outer [(1 term inner)] = $A_{L,0}$

$\therefore\ A_{L,0} = A_{out,0} = eb$

using BC at $x = 1$, noting there is no boundary layer there

Note This gives $\lim_{x\to 0} y_{out,0}(x) = \lim_{x_L\to\infty} y_L(x_L)$ as previously observed

Example 2^{nd} order matching

LHS. 2 term outer $= A_{out,0}e^{-x} + \varepsilon\left(A_{out,1}e^{-x} - A_{out,0}\,x e^{-x}\right)$

$= ebe^{-\varepsilon x_L} + \varepsilon\left(A_{out,1}e^{-\varepsilon x_L} - eb\varepsilon x_L e^{-x_L\varepsilon}\right)$

$= eb - \varepsilon eb x_L + \varepsilon A_{out,1} + O(\varepsilon^2)$

RHS 2 term inner $= A_{L,0} + (a - A_{L,0})e^{-x_L} + \varepsilon\left(A_{L,1} - A_{L,1}e^{-x_L} - A_{L,0}x_L + (a-A_{L,0})x_L e^{-x_L}\right)$

$= eb + (a-eb)e^{-x/\varepsilon} + \varepsilon\left(A_{L,1} - A_{L,1}e^{-x/\varepsilon} - eb\,x/\varepsilon + (a-eb)x\,\varepsilon e^{-x/\varepsilon}\right)$

$= eb + \varepsilon(A_{L,1}) - ebx$ + exponentially small terms.

Noting $\varepsilon x_L = x$, we have $A_{L,1} = A_{out,1} = eb$

↑ using BC at $x=1$, noting there is no boundary layer there

$$\therefore \quad y_{out}(x) = ebe^{-x} + \varepsilon eb(1-x)e^{-x} + \cdots$$

$$y_L(x_L) = eb + (a-eb)e^{-x_L} + \varepsilon\left(eb\left(1-e^{-x_L}\right) - eb\,x_L + (a-eb)x_L e^{-x_L}\right) + \cdots$$

Exercise repeat for 1 term inner [(2 terms outer)] = 2 terms outer [(1 term inner)]

Warning Treat Logarithmic terms as $O(1)$ in Van Dyke's matching rule due to their size relative to powers.

Composite Expansion

Aim To combine inner and outer expansions to obtain a uniformly valid expansion (for plotting etc)

$$y_{composite} = (p \text{ terms outer}) + (p \text{ terms inner}) - \underbrace{p \text{ terms inner}\left[(p \text{ terms outer})\right]}_{p \text{ terms outer}\left[(p \text{ terms inner})\right] \text{ by Van Dyke.}} \quad p \in \mathbb{N}$$

Subtract p terms inner [(p terms outer)] as it has been counted twice in the overlap region.

Example

p=1

$$y_{composite} = y_{out,0}(x) + y_{L,0}(x/\varepsilon) - 1 \text{ term inner}\left[(1 \text{ term outer})\right]$$

$$= ebe^{-x} + eb + (a-eb)e^{-x/\varepsilon} - eb$$

$$= ebe^{-x} + (a-eb)e^{-x/\varepsilon}$$

p=2

$$y_{composite} = y_{out,0}(x) + \varepsilon y_{out,1}(x) + y_{L,0}(x/\varepsilon) + \varepsilon y_{L,1}(x/\varepsilon) - 2 \text{ term inner}\left[(2 \text{ term outer})\right]$$

$$= ebe^{-x} + \varepsilon \cdot eb \cdot (1-x)e^{-x}$$
$$+ eb + (a-eb)e^{-x/\varepsilon} + \varepsilon\left(eb(1-e^{-x/\varepsilon}) - ebx/_{\varepsilon} + (a-eb)\frac{x}{\varepsilon}e^{-x/\varepsilon}\right)$$
$$- eb + ebx - \varepsilon eb$$

$$= ebe^{-x} + (a-eb)(1+x)e^{-x/\varepsilon} - \varepsilon eb(1-x)e^{-x} - eebe^{-x/\varepsilon}$$

Choice of rescaling, revisited

In left hand boundary layer, began with scaling $x = \varepsilon^{\alpha} x_L$, $y(x) = y_L(x_L)$.

$$\varepsilon^{1-2\alpha}\frac{d^2 y_L}{dx_L^2} + \varepsilon^{-\alpha}\frac{dy_L}{dx_L} + y_L = 0$$

$\alpha = 0$		Balance		Outer Solution
$0 < \alpha < 1$		Dominant		Overlap region
$\alpha = 1$	Balance			Inner Solution
$\alpha > 1$	Dominant			Sub-inner

The inner and outer solutions can be matched as they share a common term, which is dominant in the overlap region

There are two dominant balances

$$\alpha = 0 \text{ (outer)} \quad \text{and} \quad \alpha = 1 \text{ (inner)}$$

These correspond to distinguished limits in which $x = \mathrm{ord}(1)$ and $x = \mathrm{ord}(\varepsilon)$ respectively.

5.2 Where is the boundary layer?

- For a non-trivial boundary layer, the inner solution decays on approaching the outer region.

Saw this previously in the example

- New example

$$\varepsilon y'' + p(x) y' + q(x) y = 0 \qquad 0 < x < 1$$

$$y(0) = A \quad y(1) = B \qquad 0 < \varepsilon \ll 1$$

$$p, q \text{ smooth}; \; p(x) > 0$$

RH boundary layer

Let $x = 1 + \delta \hat{x} \quad y(x) = y_R(\hat{x})$

as ever $'$ is derivative wrt argument.

$$\frac{\varepsilon}{\delta^2} y_R'' + \underbrace{p(1+\delta\hat{x})}_{p(1)+O(\delta)} \frac{1}{\delta} y_R' + \underbrace{q(1+\delta\hat{x})}_{q(1)+O(\delta)} y_R = 0$$

Only balance with y_R'' is between 1^{st} & 2^{nd} terms $\therefore$ $\boxed{\varepsilon = \delta}$

$$\therefore \quad y_R'' + \left[p(1) + \varepsilon\hat{x}p'(1) + \cdots\right] y_R' + \varepsilon\left[q(1) + \varepsilon\hat{x}q'(1) + \cdots\right] y_R = 0$$

With $y_R(\hat{x}) \sim y_{R,0} + \varepsilon y_{R,1} + \cdots$

$\underline{O(\varepsilon^0)}$ $\quad y_{R,0}'' + p(1) y_{R,0}' = 0$

$\therefore\ y_{R,0}(\hat{x}) = J + Ke^{-p(1)\hat{x}}$

Matching $y_{R,0}(-\infty)$ with outer implies $K \equiv 0$, as we have exponential blow up.

$\therefore\ y_{R,0}(\hat{x}) = A$ and no rapid variation in boundary layer

$\therefore$ No boundary layer required.

LH Boundary layer

$y(x) = y_L(\hat{x})$, $x = \varepsilon\hat{x}$, $\quad y_{L,0} = M + Ne^{-p(0)\hat{x}}$

Possible to match outer solution without $N \equiv 0$, as $y_{L,0}(\infty)$ finite $\therefore$ Can have boundary layer, illustrating above statement.

Example

$$\varepsilon^2 f'' + 2f(1-f^2) = 0 \qquad |x|<1 \quad f(\pm 1) = \pm 1.$$

- Outer solution one of $f = 0, 1, -1$.
- Near LH boundary $f = -1$ OK; similarly $f = +1$ near RH boundary
- Boundary layer in interior

Let $x = x_0 + \varepsilon X \qquad f(x) = F(X)$

$$\therefore\ F'' + 2F(1-F^2) = 0 \qquad X \in (-\infty, \infty)$$

$$F \to \pm 1 \qquad \text{as } X \to \pm\infty$$

Solution $\quad F(X) = \tanh(X - X_*)$ ← constant.

Note x_0, X_* undetermined.

By symmetry $f(x) = -f(-x)$ as both satisfy ODE.

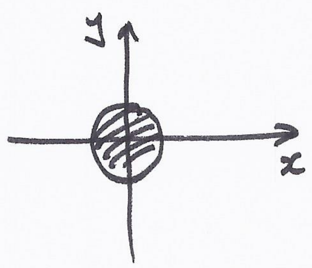
$\therefore f(0) = -f(0) \quad \therefore x_0 = X_* = 0$

$\therefore f = \tanh(x/\varepsilon)$ Agrees with exact solution

Position of transition layer exponentially sensitive to BCs. Can be analysed with WKBJ method, but beyond scope of course.

5.3 Boundary Layers in PDES

Example (2D) $\quad \underline{u} \cdot \nabla T = \varepsilon \nabla^2 T \quad$ for $r^2 = x^2 + y^2 > 1$



with $T = 1$ on $r = 1$ and $T \to 0$ as $r \to \infty$

$\underline{u} = \nabla \varphi, \quad \varphi = (r + 1/r)\cos\theta = x + \dfrac{x}{x^2+y^2}$

Outer $\quad T \sim T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \dots \quad$ as $\varepsilon \to 0^+$ with $r = \text{ord}(1)$.

$O(\varepsilon^0) \quad \underline{u} \cdot \nabla T_0 = 0, \; T_0 \to 0$ as $r \to \infty$, $\underline{r > 1 \text{ (Outer)}}$

on a curve with $\frac{d\underline{r}}{ds} = \underline{u}$, $\frac{dT_0}{ds} = \nabla T_0 \cdot \frac{d\underline{r}}{ds} = \nabla T_0 \cdot u = 0$

curve arclength

For $r > 1$ $\quad \frac{dx}{ds} = \frac{\partial \varphi}{\partial x} = 1 + \frac{y^2 - x^2}{(x^2+y^2)^2} = 1 - \frac{\cos 2\theta}{r^2} > 0$

$\therefore$ For $r > 1$, all such curves go to infinity, where $T_0 = 0$

$\therefore T_0 = 0$ as T_0 invariant on these curves.

__Inner__

$$\left(1 - \frac{1}{r^2}\right)\cos\theta \; T_r - \left(1 + \frac{1}{r^2}\right)\frac{\sin\theta}{r} T_\theta = \varepsilon\left(T_{rr} + \frac{1}{r}T_r + \frac{1}{r^2}T_{\theta\theta}\right)$$

Let $r = 1 + \delta(\varepsilon)\rho$ $\quad T(r,\theta) = \hat{T}(\rho,\theta)$ with

$\delta \to 0^+$, $\rho = \text{ord}(1)$ as $\varepsilon \to 0^+$.

$$\therefore \left(1 - \frac{1}{(1+\delta\rho)^2}\right)\frac{\cos\theta}{\delta}\hat{T}_\rho - \left(1 + \frac{1}{(1+\delta\rho)^2}\right)\frac{\sin\theta}{1+\delta\rho}\hat{T}_\theta$$

$$= \varepsilon/\delta^2 \, \hat{T}_{\rho\rho} + \frac{\varepsilon}{\delta(1+\delta\rho)}\hat{T}_\rho + \frac{\varepsilon}{(1+\delta\rho)^2}\hat{T}_{\theta\theta}$$

OKAY

$$\therefore \left(2\delta\rho + O(\delta^2)\right)\frac{\cos\theta}{\delta}\hat{T}_\rho - \cancel{\left(2 + O(\delta)\right)\sin\theta\,\hat{T}_\theta}$$

$$= \varepsilon/\delta^2 \, \hat{T}_{\rho\rho} + \cancel{\varepsilon/\delta\left(1 + O(\delta)\right)\hat{T}_\rho} + \cancel{\varepsilon\left(1 + O(\delta)\right)\hat{T}_{\theta\theta}}$$

never going to balance

$\varepsilon/\delta^2 \sim O(1)$ Let $\delta = \varepsilon^{1/2}$

$$\hat{T} = \hat{T}_0 + \varepsilon \hat{T}_1 + \cdots$$

$$2\rho\cos\theta \frac{\partial \hat{T}_0}{\partial \rho} - 2\sin\theta \frac{\partial \hat{T}_0}{\partial \theta} = \frac{\partial^2 \hat{T}_0}{\partial \rho^2}$$

BC $\hat{T}_0 = 1$ on $\rho = 0$ with $\hat{T}_0 \to 0$ as $\rho \to \infty$ to match outer.

(Leading order Outer limit of inner is Inner limit of outer $\equiv 0$).

Seek similarity solution of form $\hat{T}_0 = f(\eta)$, $\eta = \rho g(\theta)$

$$\frac{\partial \hat{T}_0}{\partial \rho} = g f' \qquad \frac{\partial^2 \hat{T}_0}{\partial \rho^2} = g^2 f'' \qquad \frac{\partial \hat{T}_0}{\partial \theta} = \rho g' f'$$

$$\therefore \quad 2\rho\cos\theta\, g(\theta) f' - 2\sin\theta\, \rho g'(\theta) f' = g^2(\theta) f''$$

$$\therefore \quad (\rho g(\theta)) f' \left[\underbrace{\frac{2\cos\theta}{g^2(\theta)} - \frac{2\sin\theta\, g'(\theta)}{g^3}}_{\text{-ve constant}} \right] = f''$$

exercise: Show this is indeed WLOG by solving for a general negative constant and confirming the same result for $\hat{T}_0$ is found

-ve constant (if not -ve f will blow up as $\rho \to \infty$)

given g>0

WLOG Set constant $= -1$

$\therefore$ Solve $2\cos\theta\, g(\theta) - 2\sin\theta\, g'(\theta) = -g^3(\theta)$

generates error function which is well understood

Let $g = \dfrac{1}{p^{1/2}}$ will convert this to a linear ODE

Find $g(\theta) = \dfrac{|\sin\theta|}{(J + \cos\theta)^{1/2}}$ is a suitable solution.

While J is the constant of integration ... < 1 means blow up ... > 1 means T=1 for theta = Pi ... makes no sense ... upstream heated ... hence choose J= 1.

Then $f'' + \eta f' = 0 \quad \therefore \quad f = J\int_\eta^\infty e^{-u^2/2}\,du + K$

With $\hat{T}_0 = f(\eta) \to 0$ as $\rho \to \infty$, i.e. $\eta \to \infty$, $K = 0$.

$\hat{T}_0(\rho = 0) = 1 \quad \therefore \quad f(0) = 1 \quad \therefore \quad f(\eta) = \sqrt{\frac{2}{\pi}}\int_\eta^\infty e^{-u^2/2}\,du$

$\therefore$ Solution to leading order is

$$T(r,\theta) = \hat{T}(\rho,\theta) = f(\rho g(\theta))$$

$$= \sqrt{\frac{2}{\pi}}\int_{\frac{(r-1)}{\varepsilon^{1/2}}\frac{|\sin\theta|}{(1+\cos\theta)^{1/2}}}^{\infty} e^{-u^2/2}\,du$$

Solution fails for $\theta \equiv 0$

as we do not satisfy BC ~~near~~ at infinity.

Boundary Layer at infinity, logs

$$(x^2y')' + \varepsilon x^2 y y' = 0 \qquad x>1,\ y(1)=0,\ y(\infty)=1$$
$$0<\varepsilon\ll 1$$

Try $y \sim y_0(x) + \varepsilon y_2(x) + \dots$ Know this expansion is incorrect a posteriori (hence the y2) ... to see why, let's try it

$O(\varepsilon^0)$ $\quad (x^2 y_0')' = 0 \quad \therefore y_0 = 1 - 1/x$ using boundary conditions.

$O(\varepsilon^1)$ $\quad (x^2 y_2')' = -x^2 y_0 y_0' = -1 + 1/x$

$\therefore$ using $y_2(1)=0$, $\quad y_2 = A(1-1/x) - \ln x - \dfrac{\ln x}{x}$

cannot satisfy $y_2(\infty)=0$ (Both $-1+1/x$ are homogeneous solutions to $(x^2 f')'=0$, hence a resonant forcing occurs)

Try $x = X/\delta_1(\varepsilon)$ $\quad y = 1+\delta_2(\varepsilon)Y(X)$ with $\delta_1, \delta_2 \to 0$, $X = \text{ord}(1)$ as $x\to\infty$

Dominant balance
$$\delta_2 \frac{d}{dX}\left(X^2\frac{dY}{dX}\right) + \frac{\varepsilon\delta_2}{\delta_1}X^2\frac{dY}{dX} + \frac{\varepsilon\delta_2^2}{\delta_1}X^2 Y\frac{dY}{dX} = 0$$

$\delta_1 = \varepsilon$, δ_2 undetermined

small "oh"

Let $Y(X) = Y_0(X) + o(1)$

$$\frac{d}{dX}\left(X^2\frac{dY_0}{dX}\right) + X^2\frac{dY_0}{dX} = 0$$

$$Y_0(X) = B\int_X^\infty \frac{e^{-s}}{s^2}\,ds \quad \text{noting } Y_0(\infty)=0$$

exercise

$\underbrace{\text{Splitting range of integral}}$, $\quad Y_0(X) = B\left[1/X + \ln X + O(1)\right]$ as $X\to 0^+$.

Need this limit for matching

Intermediate variables $\quad \hat{x} = \varepsilon^\alpha x = \varepsilon^{\alpha-1}X$

$$y = 1 + \delta_2 Y \sim 1 + \delta_2 B\left[\frac{\varepsilon^{\alpha-1}}{\hat{x}} + \ln(\varepsilon^{1-\alpha}\hat{x}) + \cdots\right]$$ for "inner"

$y \sim 1 - \frac{\varepsilon^\alpha}{\hat{x}}$ for outer $\quad \therefore$ Let $\delta_2 = \varepsilon$, $B = 1$

$$\therefore \quad 1 + \delta_2 Y \sim 1 - \frac{\varepsilon^\alpha}{\hat{x}} - \varepsilon \ln(\varepsilon^{1-\alpha}\hat{x}) + \cdots$$

$$\sim 1 - \frac{\varepsilon^\alpha}{\hat{x}} + \underbrace{(1-\alpha)\varepsilon \ln 1/\varepsilon}_{\text{next term scales with } \varepsilon \ln 1/\varepsilon} - \underbrace{\varepsilon \ln \hat{x}}_{\text{then scale with } \varepsilon}$$

$\therefore$ We should have written $\quad y \sim y_0(x) + \varepsilon \ln 1/\varepsilon \, y_1(x) + \varepsilon y_2(x) + \cdots$

for the outer ...

Now we can match ....

$(x^2 y_1')' = 0 \qquad y_1(x) = C(1 - 1/x)$ using $y_1(1) = 0$.

$$\therefore \quad y \sim 1 - \frac{\varepsilon^\alpha}{\hat{x}} + \varepsilon \ln\frac{1}{\varepsilon}\, C\left(1 - \varepsilon^\alpha/\hat{x}\right)$$
$$+ \varepsilon\left[A\left(1 - \varepsilon^\alpha/\hat{x}\right) - \ln(\varepsilon^{-\alpha}\hat{x}) - \varepsilon^\alpha/\hat{x}\, \ln(\varepsilon^{-\alpha}\hat{x})\right] + \cdots$$ in intermediate region

for the outer ...

$$\sim 1 - \frac{\varepsilon^\alpha}{\hat{x}} + \left(\varepsilon \ln\frac{1}{\varepsilon}\right)\left[C - \alpha\right] + \cdots$$

$\therefore \quad 1 - \alpha = C - \alpha$ and $C = 1$

can now match the inner at leading order

$$\therefore \quad y \sim (1 - 1/x) + \varepsilon \ln 1/\varepsilon\, (1 - 1/x) + O(\varepsilon)$$

5.2.9

Expansion sequence $1, \varepsilon \ln 1/\varepsilon, \varepsilon, \varepsilon^2 \ln 1/\varepsilon, \varepsilon^2 (\ln 1/\varepsilon)^2, \varepsilon^2, \ldots$

Van Dyke rule works only if $(\ln 1/\varepsilon)$ treated as $O(1)$.

but we've used ln(1/epsilon)>>1 in the expansions, so not self-consistent, and thus not satisfactory.

6 Multiple Scales

Van der Pol oscillator

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0$$

with $\quad x = 1, \ \dot{x} = 0 \ \text{at } t = 0$

Let $x \sim x_0(t) + \varepsilon x_1(t) + \cdots$

With regular perturbation expansion

$$x_0(t) = \cos t$$

$$\ddot{x}_1 + x_1 = (1 - x_0^2)\dot{x}_0 \quad \text{with} \quad x_1(0) = \dot{x}_1(0) = 0.$$

$$\therefore \ \ddot{x}_1 + x_1 = (1-\cos^2 t)(-\sin t) = \tfrac{1}{4}\sin 3t - \underbrace{\tfrac{3}{4}\sin t}$$

Will generate resonant terms

$$x_1 = \tfrac{3}{8}(t\cos t - \sin t) - \tfrac{1}{32}(\sin 3t - 3\sin t)$$

$$\therefore \ x \sim \cos t + \varepsilon\left[\underbrace{\tfrac{3}{8}t\cos t} + \cdots\right] + O(\varepsilon^2)$$

Perturbation expansion breaks down when $t \sim O(1/\varepsilon)$ as x_1 as large as x_0

Long timescales allow corrections to accumulate.

Two timescales

$\tau = t$ - fast timescale of oscillation

$T = \varepsilon t$ - slow timescale of amplitude drift

- Look for a solution of the form

$$x(t,\varepsilon) = x(\tau, T, \varepsilon)$$

treating τ, T as independent.

- $$\frac{d}{dt} = \frac{d\tau}{dt}\frac{\partial}{\partial \tau} + \frac{\partial T}{\partial t}\frac{d}{dT} = \frac{\partial}{\partial \tau} + \varepsilon\frac{\partial}{\partial T}$$

Converting ODE to PDE but freedom in T dependence used to our advantage.

$$\therefore \ddot{x} = x_{tt} = (\partial_\tau + \varepsilon\partial_T)(\partial_\tau + \varepsilon\partial_T)x = x_{\tau\tau} + 2\varepsilon x_{\tau T} + \varepsilon^2 x_{TT}$$

$$\therefore 0 = x_{tt} + \varepsilon(x^2-1)x_t + x$$

$$= x_{\tau\tau} + 2\varepsilon x_{\tau T} + \varepsilon^2 x_{TT} + \varepsilon(x^2-1)(x_\tau + \varepsilon x_T) + x$$

<u>Expand</u> $\quad x(\tau, T, \varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \cdots$

<u>$O(\varepsilon^0)$</u>

$$\begin{cases} x_{0\tau\tau} + x_0 = 0 \\ x_0(0) = 1, \quad x_{0\tau}(0) = 0 \end{cases} \qquad \therefore x_0(\tau, T) = R(T)\cos(\tau + \theta(T))$$

<u>ICs</u> $\quad \underbrace{R(0) = 1, \quad \theta(0) = 0.}_{\text{No other constraints on } R(T),\ \Theta(T) \text{ at this point.}}$

<u>$O(\varepsilon^1)$</u>

$$x_{1\tau\tau} + x_1 = -x_{0\tau}(x_0^2 - 1) - 2x_{0\tau T}$$

$$= 2R\theta_T\cos(\tau+\theta) + \left(2R_T + \frac{R^3}{4} - R\right)\sin(\tau+\theta)$$

Will generate resonance

$$+ \frac{R^3}{4}\underbrace{\sin 3(\tau+\theta)}_{\text{will not generate resonance}}$$

with $\quad x_1(0) = 0, \quad x_{1\tau}(0) = -x_{0T}(0) = -R_T(0)$

6.3/

$\therefore$ Let $R(T)\underset{\sim}{\theta_T}(T) = 0 = \left(2R_T + R^3/4 - R\right)$ } Known as "**secular**" conditions – required to avoid resonance

$\therefore \theta_0 = \text{const}$, with $\theta(0) = 0 \therefore \theta = 0$

$$\frac{dR}{dT} = \tfrac{1}{2}\left[R - R^3/4\right] \quad \text{with } R(0) = 1 \therefore R = \frac{2}{(1+3e^{-T})^{1/2}}$$

$$\therefore x(t,\varepsilon) = x(\tau, T, \varepsilon) = \underbrace{\frac{2}{(1+3e^{-\varepsilon t})^{1/2}}}_{\text{Amplitude} \to 2 \text{ as } t\to\infty,\ \varepsilon \text{ fixed.}} \cos t + O(\varepsilon)$$

<u>Higher order</u>

We find $x_1 = -R^3/32 \sin 3\tau + S(T)\sin(\tau + \varphi(T))$

To find $S(T), \varphi(T)$ resonant terms are suppressed for x_2 via secular conditions.

However to suppress resonance we must expand with a slow-slow timescale $T_2 = \varepsilon^2 t$.

<u>To see this, simpler example (do not lecture)</u>

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0$$

$$x = A\underset{\sim}{e^{-\varepsilon t}}\cos\left(\sqrt{1-\varepsilon^2}\,t + B\right)$$

amplitude drift $t \sim O(1/\varepsilon)$

phase drift on $t \sim O(1/\varepsilon^2)$

$\therefore$ Let $\tau = t$, $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t$

$$d/dt = \partial/\partial\tau + \varepsilon\,\partial/\partial T_1 + \varepsilon^2\,\partial/\partial T_2$$

} and expand as above

Similarly for higher orders of the Van der Pol oscillator

NB often presented via a complex representation

eg. Van der Pol

conjugate

$$x_0 = R(T)\cos(\tau + \theta(T)) = \tfrac{1}{2}\left(Ae^{i\tau} + \bar{A}e^{-i\tau}\right)$$

$$A = Re^{i\theta}$$

At $O(\varepsilon^1)$

$$x_{1\tau\tau} + x_1 = -2x_{0\tau T} - (x_0^2 - 1)x_{0\tau}$$

$$= -i\left(A_T e^{i\tau} - \bar{A}_T e^{-i\tau}\right) - \left[\tfrac{1}{4}\left(Ae^{i\tau} + \bar{A}e^{-i\tau}\right)^2 - 1\right] \cdot \tfrac{i}{2}\left[Ae^{i\tau} - \bar{A}e^{-i\tau}\right]$$

$$= \left[-i\left(A_T - \frac{A(4-|A|^2)}{8}\right)e^{i\tau} + \left(\begin{matrix}\text{Complex}\\ \text{Conjugate}\end{matrix}\right)\right] + \left[\begin{matrix}\text{Non}\\ \text{secular}\\ \text{terms}\end{matrix}\right]$$

$\therefore$ Suppressing resonant terms, $e^{\pm i\tau}$

$$A_T = \frac{A}{8}\left(4 - |A|^2\right) \qquad \text{with } A = Re^{i\theta}$$

$$\therefore \quad R_T e^{i\theta} + iR\theta_T e^{i\theta} = \frac{Re^{i\theta}}{8}\left(4 - R^2\right)$$

$$\therefore \quad R_T + iR\theta_T = R/8\left(4 - R^2\right) \quad \left\{\begin{matrix}\text{real \& imag}\\ \text{parts}\end{matrix}\right\} \quad \begin{matrix} R\theta_T = 0 \\ R_T = R/8\,(4 - R^2)\end{matrix}$$

as before

Note sometimes the slow variable, τ, is given the same label as the physical variable t, so that

$$x_0 = R(T)\cos(t + \theta(T)) = \tfrac{1}{2}\left(Ae^{it} + \bar{A}e^{-it}\right) \text{ above etc.}$$

Homogenization

Example $\frac{d}{dx}\left(D(x, x/\varepsilon)\frac{du}{dx}\right) = f(x) \qquad 0<x<1 \qquad (+)$

$u(0) = a, \quad u(1) = b \qquad a, b \in \mathbb{R}^+$

D, f are smooth, with $0 < D_-(x) < D(x, \overset{X = x/\varepsilon}{X}) < D_+(x)$, with $D_\pm$ continuous.

Question Can (+) be approximated by $d/dx\left(\bar{D}(x)\frac{du}{dx}\right) = f(x)$

$u(0) = a, \; u(1) = b$

for an averaged function $\bar{D}(x)$ ← does not contain fast ε variation.

Multiple Scales Let $u(x, \varepsilon) = \underset{\text{not relabelling as separate variable}}{u(x, X, \varepsilon)}$ with $\underbrace{X = x/\varepsilon}_{\text{fast variable}}$

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial X}$$

$$\therefore \left(\partial_x + \tfrac{1}{\varepsilon}\partial_X\right)\left[D(x, X)\left(\partial_x + \tfrac{1}{\varepsilon}\partial_X\right)u\right] = f(x)$$

$$\therefore \left(\varepsilon\partial_x + \partial_X\right)\left[D(x, X)\left(\varepsilon\partial_x + \partial_X\right)u\right] = \varepsilon^2 f(x)$$

Let $u \sim u_0(x, X) + \varepsilon u_1(x, X) + \cdots$

{ Also assume $u_0, u_1, u_2, \ldots$ bounded for ALL X }

$O(\varepsilon^0)$ $\quad \left(D(x, X)\, u_{0X}\right)_X = 0$

$O(\varepsilon^1)$ $\quad \left(D(x, X)[u_{1X} + u_{0x}]\right)_X + \left(D(x, X)\, u_{0X}\right)_x = 0$

$O(\varepsilon^2)$ $\quad \left(D(x, X)\left[u_{2X} + u_{1x}\right]\right)_X + \left(D(x, X)\left[u_{1X} + u_{0x}\right]\right)_x = f(x)$

$O(\varepsilon^0)$

$$D u_{0X} = c_1(x)$$

$$\therefore \quad u_0 = c_2(x) + c_1(x) \underbrace{\int_0^X \frac{ds}{D(x,s)}}_{\text{ord}(X) \text{ as } X\to\infty} \qquad c_1, c_2 \text{ arbitrary}$$

$$\text{as } \frac{1}{D_+(x)} \leq \frac{1}{D} \leq \frac{1}{D_-(x)}$$

u_0 bounded $\therefore$ $u_0 = c_2(x)$ and $c_1(x) = 0$ all x.

$\therefore$ We write $u_0 = u_0(x)$.

$O(\varepsilon^1)$

$$\left(D(x,X)\left[u_{0x} + u_{1X}\right]\right)_X = 0$$

$$D\left[u_{0x} + u_{1X}\right] = d_1(x) \qquad (++)$$

$$\therefore \quad u_1 = d_1(x) \int_0^X \frac{ds}{D(x,s)} - X u_{0x} + d_2(x)$$

Blow up as $X \to \infty$, with both ord(X).
Hence $D_H(x)$ exists.

$$\therefore \quad \text{Let} \quad d_1(x) = \left[\lim_{X\to\infty} \frac{X}{\int_0^X \frac{ds}{D(x,s)}}\right] u_{0x} \equiv D_H(x)\, u_{0x}$$

$O(\varepsilon^2)$

$$\left(D(x,X)\left[u_{2X} + u_{1x}\right]\right)_X = f(x) - d_{1x} \qquad \text{using } (++).$$

$$\therefore \quad D(x,X)\left[u_{2X} + u_{1x}\right] = e_1(x) + \left(f(x) - d_{1x}\right) X$$

$$\therefore \quad u_{2X} = \frac{e_1(x)}{D(x,X)} + \left(f(x) - d_{1x}\right)\frac{X}{D(x,X)} - u_{1x}$$

$$\therefore \quad u_2 = e_1(x) \underbrace{\int_0^X \frac{ds}{D(x,s)}}_{\text{ord}(X) \text{ as } X\to\infty} + \left(f(x) - d_{1x}\right) \underbrace{\int_0^X \frac{s}{D(x,s)}\, ds}_{\text{ord}(X^2) \text{ as } X\to\infty} - \underbrace{\int_0^X u_{1x}\, ds}_{\text{ord}(X) \text{ as } X\to\infty} + e_2(x)$$

NB $$u_1 = \left\{ \underbrace{\left[\lim_{P\to\infty} \left\{ \frac{P}{\int_0^P \frac{ds}{D(x,s)}} \right\} \int_0^X \frac{ds}{D(s,x)} \right] - X}_{\text{ord}(1) \text{ as } X\to\infty} \right\} u_{0,x} + d_2(x)$$

$$\therefore \int_0^X u_{1x}\, ds = \int_0^X \text{ord}(1)\, u_{0xx} + d_{2x}\, ds = \text{ord}(X) \quad \text{as } X\to\infty$$

$\therefore$ For $u_2(x, X)$ to be bounded, we must have

$$d_{1x} = f(x)$$

$$\boxed{\begin{aligned} &\therefore \quad \frac{d}{dx}\left(D_H(x) \frac{du_0}{dx} \right) = f(x) \qquad (*) \\ &\text{with} \quad D_H(x) = \lim_{X\to\infty} \left[\frac{X}{\int_0^X \frac{ds}{D(x,s)}} \right] \\ &u_0(0) = a, \quad u_0(1) = b \end{aligned}}$$

(*) Is a "homogenized" ODE

NB If $D(x,s)$ is periodic, say with period 1, D_H simplifies by taking $X \in \mathbb{N}$

$$D_H = \lim_{X\to\infty} \left[\frac{X}{X \int_0^1 \frac{ds}{D(x,s)}} \right] = \frac{1}{\int_0^1 \frac{ds}{D(x,s)}}$$

In higher dimensional problems, periodicity often has to be assumed to make progress

7. WKB Method

- After Wentzel, Kramers, Brillouin (1920s) Also known as WKBJ where J is for Jeffries. First used by Liouville and Green in (1830s).
- Important in semi-classical analysis of quantum mechanics.

Example

$$\left.\begin{array}{l}\varepsilon^2 y'' + y = 0 \\ 0 < \varepsilon \ll 1\end{array}\right\} \Rightarrow y = R\cos\left(x/\varepsilon + \theta\right) \quad ; R, \theta \in \mathbb{R}, \text{ consts.}$$

High frequency oscillations. What if frequency of oscillation depends on the slow scale...

Example

$$\varepsilon^2 y'' + q(x) y = 0 \qquad q(x) > 0$$
$$0 < \varepsilon \ll 1$$

Try multiple scales $\quad x = \varepsilon X \quad \therefore \quad \dfrac{d^2y}{dX^2} + q(\varepsilon X) y = 0$

treat x, X as independent

$$\frac{dy}{dX} = \frac{\partial y}{\partial X} + \frac{\partial x}{\partial X}\frac{\partial y}{\partial x} = \left(\frac{\partial}{\partial X} + \varepsilon \frac{\partial}{\partial x}\right) y$$

$$\therefore \quad y_{XX} + 2\varepsilon y_{xX} + \varepsilon^2 y_{xx} + q(x) y = 0$$

Let $y = y_0(x, X) + \varepsilon y_1(x, X) + \cdots$

<u>0^{th} order</u> $\quad y_{0XX} + q(x) y_0 = 0 \quad \therefore \quad y_0 = R(x)\cos\left(\sqrt{q(x)}\,X + \theta(x)\right)$

<u>1^{st} order</u>

$$y_{1XX} + q(x) y_1 = -2 y_{0xX}$$

$$= -2\left[R(x)\cos\left(\sqrt{q(x)}\,X + \theta(x)\right)\right]_{xX}$$

$$= +2\left[\sqrt{q(x)}\,R(x)\sin\left(\sqrt{q(x)}\,X + \theta(x)\right)\right]_{x}$$

$$= 2\frac{\partial}{\partial x}\left[\sqrt{q(x)}\,R(x)\right]\sin\left(\sqrt{q(x)}\,X+\theta(x)\right) + 2\frac{\partial}{\partial x}\left(\sqrt{q(x)}\,X+\theta(x)\right)\sqrt{q}\,R(x)\cos\left(\sqrt{q(x)}\,X+\theta(x)\right)$$

Both terms on RHS are resonant.

<u>Secular conditions</u> : $\quad \frac{\partial}{\partial x}\left(\sqrt{q(x)}\,R(x)\right) = 0 = \sqrt{q}\,R\,\frac{\partial}{\partial x}\left(\sqrt{q(x)}\,X + \theta(x)\right)$

$q > 0 \therefore$ Either $R(x) = 0$ (trivial solution obvious and useless)

or $\quad \frac{\partial}{\partial x}\left(\sqrt{q(x)}\,X + \theta(x)\right) = 0 \quad$ i.e. $\quad \frac{\partial}{\partial x}\left(\sqrt{q}\right)X + \theta'(x) = 0$

i.e. $$X = -\frac{\theta'(x)}{\partial/\partial x\left(\sqrt{q}\right)}$$ ⋇ <u>function of x cannot be equal to X for all X.</u>

Happens whenever frequency of fast oscillation drifts on a slow scale.

$\therefore$ <u>Try a WKB expansion</u> $\quad y = \exp\left[\frac{i}{\varepsilon}\varphi(x)\right]A(x,\varepsilon)$

$$y' = e^{i\varphi/\varepsilon}\left[i\varphi' A/_{\varepsilon} + A' \right]$$

$$y'' = e^{i\varphi/\varepsilon}\left[\frac{i\varphi'}{\varepsilon}\left(i\varphi' A/_{\varepsilon} + A' \right) + \left(i\varphi' A/_{\varepsilon} + A' \right)' \right]$$

$$\therefore \quad \varepsilon^2 e^{i\varphi/\varepsilon}\left[\frac{-\varphi'^2 A}{\varepsilon^2} + \frac{2i\varphi' A' \; + i\varphi'' A}{\varepsilon} + A'' \right] + q e^{i\varphi/\varepsilon} A = 0$$

$$\therefore \quad \varepsilon^2 A'' + \left\{ 2i\varepsilon\varphi' A' \right\} + \left\{ -\varphi'^2 + i\varepsilon\varphi'' + q \right\} A = 0$$

Let $A = A_0 + \varepsilon A_1 + \cdots$

$O(\varepsilon^0)$ $\quad \left\{ -\varphi'^2 + q \right\} A_0 = 0 \quad \therefore$ For $A_0 \not\equiv 0$, $\quad \underline{\underline{\varphi'^2 = q}}$

$O(\varepsilon^1)$ $\quad 2i\varphi' A_0' + i\varphi'' A_0 + \underbrace{\left\{ -\varphi'^2 + q \right\}}_{0} A_1 = 0$

$$\therefore \quad \frac{2A_0'}{A_0} + \frac{\varphi''}{\varphi'} = 0$$

$$\therefore \quad \log A_0^2 \varphi' = \text{Const} \quad \therefore \quad A_0 = \frac{\alpha_0}{(\varphi')^{1/2}} \qquad \alpha_0 \in \mathbb{C}$$

$O(\varepsilon^{n+1})$ $\quad A_{n-1}'' + 2i\varphi' A_n' + i\varphi'' A_n = 0$

$$\therefore \quad \left((\varphi')^{1/2} A_n \right)' = -\frac{1}{2i(\varphi')^{1/2}} A_{n-1}''$$

$$\therefore \quad A_n = \frac{i}{2(\varphi')^{1/2}} \int^{x} \frac{A_{n-1}''(s)}{2(\varphi')^{1/2}(s)}\, ds$$

At leading order

$$y \sim \frac{\alpha_+}{(q(x))^{1/4}} \exp\left[\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\,ds\right] + \frac{\alpha_-}{(q(x))^{1/4}} \exp\left[-\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\,ds\right]$$

In principle can go to higher orders as A_n generally known.

$\alpha_\pm \in \mathbb{C}$.

Method breaks down near $\varphi' = 0$, as amplitude blows up.

Fix at turning points considered later.

Example

Find eigenvalues with $\lambda \gg 1$ for $p(x)$ a positive function and

$$y'' + \lambda p(x) y = 0 \qquad 0<x<1 \qquad y(0)=0 \quad y(1)=0$$

Let $\lambda = 1/\varepsilon^2$, $0<\varepsilon \ll 1$. Then

$$\varepsilon^2 y'' + p(x) y = 0$$

WKB Let $y = e^{i\varphi/\varepsilon} A(x,\varepsilon) \sim e^{i\varphi/\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n A_n(x)$.

$O(\varepsilon^0)$ $\quad \varphi'^2 = p \quad \therefore \varphi' = \pm\sqrt{p(x)} \quad \therefore \varphi = \pm \int_0^x p(s)^{1/2} ds$

const of integration absorbed into A_0.

$O(\varepsilon^1)$ $\quad 2\varphi' A_0' + \varphi'' A_0 = 0 \qquad \therefore A_0 = \dfrac{\text{const}}{(p(x))^{1/4}}$

Two lin. independent solutions

$$y_+ \sim A_0 e^{i\varphi/\varepsilon} \qquad y_- \sim A_0 e^{-i\varphi/\varepsilon}$$

General solution, at leading orders:

$$y \sim \alpha A_0(x) \cos\left(\varphi(x)/\varepsilon\right) + \beta A_0(x) \sin\left(\frac{\varphi(x)}{\varepsilon}\right)$$

$$\alpha, \beta \in \mathbb{R}.$$

$y(0)=0 \quad \therefore \alpha = 0$

small "oh"

$y(1)=0$ satisfied at leading order only if $\beta A_0(1) \sin\left(\frac{\varphi(1)}{\varepsilon}\right) = o(1)$

We have $A_0(1) \neq 0$, $\beta > 0$ for a non-trivial solution

$$\therefore \quad \varphi(1) \sim n\pi\varepsilon \quad \text{as } \varepsilon \to 0.$$

$$\therefore \frac{1}{\sqrt{\lambda_n}} = \varepsilon_n \sim \frac{\varphi(1)}{n\pi} = \frac{1}{n\pi}\int_0^1 \sqrt{p(x)}\,dx$$

n^{th} eigenvalue

$$\therefore \lambda_n \sim \left(\frac{n\pi}{\int_0^1 \sqrt{p(x)}\,dx} \right)^2 \quad \text{as } n \to \infty$$

Example Semi-Classical Quantum. Turning Points.

The non-dimensional steady state Schrödinger equation for the even wave-functions of the simple harmonic oscillator is given by

$$\psi'' - x^2\psi = -E\psi$$
$$\psi \to 0 \text{ as } x \to \infty, \quad \psi'(0) = 0.$$

Find the large, $E \gg 1$, energy eigenvalues.

Let $y = \psi$. $x = \bar{x}/\sqrt{\varepsilon}$ with $\varepsilon = 1/E$. Then, dropping bars,

$$\varepsilon^2 y'' + (1-x^2)y = 0$$
$$y(\infty) = 0, \; y'(0) = 0, \quad 0 < \varepsilon \ll 1.$$

Let $y = e^{i\varphi/\varepsilon} A(x,\varepsilon) \sim e^{i\varphi/\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n A_n(x)$

WKB $\quad O(\varepsilon^0) \quad \varphi' = \pm\sqrt{1-x^2} \quad\quad O(\varepsilon^1) \quad A_0 = \dfrac{\text{const}}{(1-x^2)^{1/4}}$

Hence

For $0<x<1$, $$y \sim \frac{M_0}{(1-x^2)^{1/4}} e^{\frac{i}{\varepsilon}\int_0^x \sqrt{1-s^2}\,ds} + \frac{N_0}{(1-x^2)^{1/4}} e^{-\frac{i}{\varepsilon}\int_0^x \sqrt{1-s^2}\,ds}$$

$$\sim \frac{P_0}{(1-x^2)^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{1-s^2}\,ds\right)$$ using $y'(0)=0$

For $x>1$ $$y \sim \frac{Q_0}{(x^2-1)^{1/4}} e^{-\frac{1}{\varepsilon}\int_1^x \sqrt{s^2-1}\,ds}$$ using $y(\infty)=0$

However, these breakdown near $x \simeq 1$ as $\varphi'(1)=0$.

Resolve using matched asymptotics

Inner region around $x=1$ Let $x = 1+\delta_1(\varepsilon)X$ $Y(X) = \delta_2(\varepsilon)\,y(x)$

$$\frac{\varepsilon^2}{\delta_1^2}\frac{d^2Y}{dX^2} + \Big(1-\underbrace{(1+2\delta_1 X + \delta_1^2 X^2)\Big)Y}_{2\delta_1 XY + \delta_1^2 X^2 Y} = 0$$

Dominant balance when $2\delta_1^3 = \varepsilon^2$ $\therefore$ Let $\delta_1 = \dfrac{\varepsilon^{2/3}}{2^{1/3}}$. δ_2 undetermined as yet small "oh"

With $Y = Y_0(X) + o(1)$

$$\frac{d^2Y_0}{dX^2} - XY_0 = 0 \quad \therefore \quad Y_0 = R_0 \, Ai(X) + S_0 \, Bi(X)$$ where Ai, Bi are Airy functions.

<u>Airy Functions</u>

$$Ai(X) = \frac{1}{\pi}\int_0^\infty \cos\left(t^3/3 + Xt\right) dt \sim \frac{1}{2\sqrt{\pi}\,X^{1/4}} e^{-2/3 X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}\,(-X)^{1/4}} \sin\left(2/3\,(-X)^{3/2} + \pi/4\right) \quad \text{as } X \to -\infty.$$

$$Bi(X) = \frac{1}{\pi}\int_0^\infty \exp\left(-t^3/3 + Xt\right) dt \sim \frac{1}{\sqrt{\pi}\,X^{1/4}} e^{2/3 X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}\,(-X)^{1/4}} \cos\left(2/3\,(-X)^{3/2} + \pi/4\right) \quad \text{as } X \to -\infty$$

<u>Matching</u> Inner $(X \to \infty)$ with RH outer $(x \to 1^+)$

$S_0 = 0$ else Y_0 blows up as $X \to \infty$.

On matching everything scales with $\frac{1}{X^{1/4}} e^{-2/3 X^{3/2}}$ whether using Van Dyke or intermediate region. Naively one gets simply $0=0$. Thus, on matching, insist the coefficients in front of $\frac{1}{X^{1/4}} e^{-2/3 X^{3/2}}$ match.

<u>Matching</u> (intermediate variable)

Let $x - 1 = \delta_1^{\beta} \hat{x} = \delta_1 X \qquad (0<\beta<1)$ with $\hat{x} = \text{ord}(1)$, $x \to 1$, $X \to \infty$, $\hat{x} > 0$.

$$Y_0 = R_0 \, Ai\left(\frac{\hat{x}}{\delta_1^{1-\beta}}\right) \sim \frac{R_0}{2\sqrt{\pi}} \frac{(\delta_1^{1-\beta})^{1/4}}{\hat{x}^{1/4}} \exp\left[-2/3 \cdot \frac{1}{(\delta_1^{1-\beta})^{3/2}} \hat{x}^{3/2}\right]$$

$$y \sim \frac{Q_0}{[(x-1)(x+1)]^{1/4}} \exp\left[-1/\varepsilon \int_1^x \sqrt{s^2-1}\, ds\right]$$

$s^2 - 1 = (s-1)(s+1)$, $s = 1 + \eta$

$$\int_1^x \sqrt{s^2-1}\, ds = \int_0^{x-1} \eta^{1/2} 2^{1/2} \sqrt{1+\eta/2}\, d\eta$$

$$= \sqrt{2} \cdot 2/_3 (x-1)^{3/2} + \dots$$

$$= \frac{2\sqrt{2}}{3} \delta_1^{3\beta/2} \hat{x}^{3/2} + \dots$$

$$\therefore 1/\varepsilon \int_1^x \sqrt{s^2-1}\, ds = \frac{1}{(2^{1/3}\delta_1)^{3/2}} \frac{2\sqrt{2}}{3} \delta_1^{3\beta/2} \hat{x}^{3/2} + \dots$$

$$\therefore\ y \sim \frac{Q_0}{2^{1/4}\delta_1^{\beta/4}\,\hat{x}^{1/4}} \exp\left[-\frac{2}{3}\frac{1}{(\delta_1^{1-\beta})^{3/2}}\hat{x}^{3/2}\right] + \cdots$$

$$\therefore\ Y = \delta_2 y \sim \frac{Q_0\,\delta_2(\varepsilon)}{2^{1/4}(\delta_1)^{\beta/4}\hat{x}^{1/4}} \exp\left[-{}^2\!/_3\frac{1}{(\delta_1^{1-\beta})^{3/2}}\hat{x}^{3/2}\right] + \cdots \sim \frac{R_0\delta_1^{1/4}}{2\sqrt{\pi}}\frac{1}{\delta_1^{\beta/4}}\frac{1}{\hat{x}^{1/4}}\exp\left[-\frac{2}{3}\frac{\hat{x}^{3/2}}{(\delta_1^{1-\beta})^{3/2}}\right]$$

$$\therefore\ \delta_2 = \delta_1^{1/4} = \left(\varepsilon^{2/3}/2^{1/3}\right)^{1/4} = \frac{1}{2^{1/12}}\varepsilon^{1/6} \quad \text{and} \quad Q_0 = \frac{1}{2^{3/4}\sqrt{\pi}}R_0$$

Matching inner ($X \to -\infty$) with LH outer ($x \to 1^-$).

Let $x - 1 = \delta_1^{\gamma}\hat{x} = \delta_1 X$ $\quad (0<\gamma<1)$ with $\hat{x} = \mathrm{ord}(1)$, $x \to 1$, $X \to -\infty$, $\hat{x} < 0$.

$$Y_0 = R_0 \mathrm{Ai}\left(\frac{\hat{x}}{\delta_1^{1-\gamma}}\right) \sim \frac{R_0(\delta_1)^{\frac{1-\gamma}{4}}}{\sqrt{\pi}(-\hat{x})^{1/4}} \sin\left(\frac{2}{3}(-\hat{x})^{3/2}\frac{1}{(\delta_1^{1-\gamma})^{3/2}} + \frac{\pi}{4}\right)$$

$$y \sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \cos\left(\pi/4\varepsilon - \frac{1}{\varepsilon}\int_x^1 \sqrt{1-s^2}\,ds\right) \quad \text{using } \int_0^1\sqrt{1-s^2}\,ds = \pi/4$$

$$\therefore \; y \sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\,\delta_1^{\gamma/4}} \cos\left(\frac{\pi}{4\varepsilon} - \frac{1}{\varepsilon}\cdot\frac{2\sqrt{2}}{3}(1-x)^{3/2} + \ldots\right)$$

← Substituting $s = 1-\eta$ in integral and using $\sqrt{1-s^2} = \eta^{1/2}\left(2^{1/2} + O(\eta)\right)$

$$\sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\,\delta_1^{\gamma/4}} \cos\left(\frac{\pi}{4\varepsilon} - \underbrace{\frac{2\sqrt{2}}{3\varepsilon}\,\delta_1^{3\gamma/2}}_{\frac{2}{3}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}}(-\hat{x})^{3/2} + \ldots\right)$$

$$\sim \delta_2^{-1} Y_0 = \frac{R_0}{\sqrt{\pi}\,(-\hat{x})^{1/4}\,\delta_1^{\gamma/4}} \sin\left(\frac{\pi}{4} + \frac{2}{3}(-\hat{x})^{3/2}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}\right)$$

With $w = \frac{2}{3}(-\hat{x})^{3/2}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}$,

$$\frac{P_0}{2^{1/4}}\cos\left(\frac{\pi}{4\varepsilon} - w\right) \sim \frac{R_0}{\sqrt{\pi}}\sin\left(\frac{\pi}{4} + w\right)$$

$$\therefore \; \frac{P_0}{2^{1/4}}\left[\cos\frac{\pi}{4\varepsilon}\cos w + \sin\left(\frac{\pi}{4\varepsilon}\right)\sin w\right] \sim \frac{R_0}{\sqrt{\pi}}\left[\sin\frac{\pi}{4}\cos w + \cos\frac{\pi}{4}\sin w\right]$$

$$\therefore \; \frac{P_0}{2^{1/4}}\cos\frac{\pi}{4\varepsilon} \sim \frac{R_0}{\sqrt{\pi}}\sin\frac{\pi}{4}\,, \qquad \frac{P_0 \sin\left(\frac{\pi}{4\varepsilon}\right)}{2^{1/4}} \sim \frac{R_0}{\sqrt{\pi}}\cos\frac{\pi}{4}$$

For $P_0, R_0 \neq 0$ $\quad \tan\left(\frac{\pi}{4\varepsilon}\right) \sim \cot\left(\pi/4\right) = 1 \quad$ as $\varepsilon \to 0$

$$\therefore \quad \frac{\pi}{4\varepsilon} \sim \frac{\pi}{4} + n\pi \qquad \text{as } n \to \infty, \quad \text{with } n \in \mathbb{N}$$

$$\therefore \quad E_n = \frac{1}{\varepsilon_n} = 1 + 4n \qquad \text{as } n \to \infty, \text{ for the energy levels.}$$

Once this holds $\quad \cos\left(\frac{\pi}{4\varepsilon}\right) \sim \cos\left(\pi/4 + n\pi\right) = \frac{1}{\sqrt{2}}(-1)^n$

$$\therefore \quad P_0 = \frac{2^{1/4}}{\sqrt{\pi}}(-1)^n R_0 = 2\,(-1)^n Q_0$$

Connection formula

$$y_n \sim \frac{Q_0}{(x^2-1)^{1/4}}\, e^{-1/\varepsilon_n \int_1^x \sqrt{s^2-1}\, ds} \qquad x > 1, \; x \not\simeq 1$$

$$\sim \frac{2^{1/12}}{\varepsilon^{1/6}} \cdot 2^{3/4} \sqrt{\pi}\, Q_0 \,\mathrm{Ai}\left(\frac{2^{1/3}(x-1)}{\varepsilon_n^{2/3}}\right) \qquad x \simeq 1$$

$$\sim \frac{2(-1)^n Q_0}{(1-x^2)^{1/4}} \cos\left(\frac{1}{\varepsilon_n}\int_0^x \sqrt{1-s^2}\, ds\right) \quad \begin{matrix} x<1 \\ x \not\simeq 1 \end{matrix}, \quad \varepsilon_n = \frac{1}{1+4n}, \quad n \gg 1.$$