Part A Numerical Analysis, Hilary 2018. Problem Sheet 2

Note: some questions are marked optional; its up to you and your tutor if you submit these.

1. Newton–Cotes Quadrature: Find the approximation to the integral

$$\int_0^1 \frac{1}{x+1}\,\mathrm{d}x$$

using the trapezium Rule and Simpson's Rule.

2. Explicitly derive Simpson's Rule from its definition in terms of the quadratic Lagrange interpolating polynomial.

3. (Optional) Find the Newton–Cotes quadrature rule based on exact integration of the cubic Lagrange interpolating polynomial. [Hint: you may find it helpful to consider symmetries and the substitution $x = x_0 + th$.]

4. (Optional)

Noting that for $b > a$, and any function f continuous on $[a, b]$,

$$\min_{x\in[a,b]} f(x) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \max_{x\in[a,b]} f(x),$$

use the Intermediate Value Theorem to show that $\exists \eta \in (a, b)$ satisfying

$$\int_a^b f(x)\,\mathrm{d}x = (b-a)f(\eta).$$

Thus if $G'(x) = g(x) \ge 0$ for $x \in [a, b]$, prove that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(\eta)\int_a^b g(x)\,\mathrm{d}x$$

for some $\eta \in (a, b)$. [Note $dG = G'(x)\,\mathrm{d}x$.]

5. (Optional)

Using the Integral Mean Value Theorem, show that

$$\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{2}[f(b) + f(a)] = -\frac{1}{12}(b-a)^3 f''(\eta) \text{ for some } \eta \in (a, b).$$

Hence show that the trapezium rule always overestimates integrals for functions satisfying $f''(x) \ge 0$. Explain geometrically why this is reasonable.

6. Show that Simpson's Rule *exactly* integrates any cubic polynomial on an interval $[a, b]$.

7. Estimate how many equal length intervals $[0, 2]$ should be broken into in order that $f(x)$ be integrated with an accuracy of 10^{-5} using the composite Simpson rule if

$$\max_{x \in [0,2]} |f^{(4)}(x)| = 1.$$

[**M**] Check how accurate or how pessimistic this estimate is by using the MATLAB function `adaptive_simpson` (available from the course website) for the function $f(x) = \cos(x)$, which you can define in MATLAB with

```
f = @(x) cos(x)
```

You may find `help adaptive_simpson` useful. Compare the numerical quadrature with the exact value of the integral, using `format long` to show more decimal places (`format short` will revert to displaying fewer decimal places).

8. [**M**] Apply `adaptive_simpson` (see question 7 above) for the following functions:

$$\int_0^{\pi/2} \cos x \, \mathrm{d}x \tag{a}$$

$$\int_{-1}^{1} |x| \, \mathrm{d}x, \qquad \text{(see } \texttt{help abs}\text{)} \tag{b}$$

$$\int_{-1}^{3/2} |x| \, \mathrm{d}x \tag{c}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, \mathrm{d}x \quad \text{approximated by} \quad \int_{-5}^{5} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, \mathrm{d}x \tag{d}$$

(since $e^{-25/2} \leq 10^{-5}$). [Recall the normal distribution from probability.] You might need to use component-wise exponentiation (`x.^2`) to specify the integrand.

Comment on what you observe in each case, in particular relating what you see to the theory covered in lectures.

9. If $x_0, x_1, \ldots, x_n$ are distinct real values, then by considering the Lagrange interpolating polynomial in the form $p_n = a_0 + a_1 x + \cdots + a_n x^n$ or otherwise prove that the square matrix

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

is nonsingular.