Part A Numerical Analysis, Hilary 2018. Problem Sheet 4

Note: some questions are marked optional; its up to you and your tutor if you submit these.

1. If $\|x\| := \sqrt{x^T x}$ is the usual (Euclidean) length of a vector $x \in \Re^n$, show that the vector Qx has the same length whenever Q is an orthogonal $n \times n$ matrix.

If we define the angle between vectors $x, y \in \Re^n$ as

$$\angle(x, y) := \cos^{-1}\left(\frac{x^T y}{\|x\|\|y\|}\right)$$

show that the angle between Qx and Qy is unchanged.

2. (*Optional*) Show that if $x, y \in \Re^n$ with $x \neq 0, y \neq 0$ then the outer product matrix, xy^T has an $n-1$–dimensional kernel. Identify the Image (also called the Range) of this matrix. Hence, identify an eigenvector which corresponds to a generally non-zero eigenvalue and give the condition under which this eigenvalue is also zero.

3. (*Optional*) If S is a real skew-symmetric matrix, so that $S^T = -S$, and assuming that $I - S$ is nonsingular, show that $(I-S)^{-1}(I+S)$ is an orthogonal matrix. (You may want to convince yourself that for a nonsingular matrix $(A^{-1})^T = (A^T)^{-1}$.)

4. Suppose that A is a square nonsingular matrix. Prove that the factors Q and R featuring in the QR factorisation of A are unique if the diagonal entries of R are all positive. How many possibilities are there if this restriction is removed?

5. By considering the QR factorisation in which the diagonal entries of R are all positive as in the question above (or otherwise), prove that any orthogonal matrix may be expressed as the product of Householder matrices.

6. (*Optional*) Determine the eigenvalues of a Householder matrix.

7. (*Optional*) If A is a real matrix with m rows and n columns (i.e. a rectangular matrix when $m \neq n$), convince yourself that the method described in lectures for QR factorisation using Householder matrices could equally well be applied for $m \neq n$. How many Householder matrices are required if $m > n$ and how many if $m \leq n$?

8. (*Optional*) If $A = QR$ is a QR factorisation of A, show that this provides a special type of triangular factorisation of $A^T A$ in which $A^T A$ is expressed as the product of an upper triangular matrix followed by its transpose.

[M] In MATLAB, `[Q, R] = qr(A)` calculates a QR factorisation of a matrix and `Q*Q'` will test if a matrix is orthogonal. Verify numerically the existence of the special triangular factorisation above for any matrix (you may want to use `randn` to construct A).

9. Show that if $x \in \Re^n$ then

$$J(1,n)J(1,n-1)\cdots J(1,3)J(1,2)\,x = \begin{bmatrix} \beta \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for some β and further prove that $\beta^2 = x^T x$.

10. Describe a sequence of $(n-1)+(n-2)+\cdots+2+1 = n(n-1)/2$ Givens matrices which when applied sequentially (as premultiplications) will reduce an $n \times n$ matrix to upper triangular form. (Hint: see question above). You have thus established another algorithm for QR factorisation. What is Q?

11. (*Optional*) If $x \in \Re^n$ show that postmultiplication of the row vector x^T by $J(i,j,\theta)$ with an appropriately chosen value of θ which you should give, will make the j^{th} entry of the resulting row vector equal to zero.

12. (*Optional*) Using the results of the above questions can you show directly (and constructively) that any matrix $A \in \Re^{n\times n}$ admits a factorisation LQ with L a lower triangular matrix and Q orthogonal. (Note if $B = QR$ then $B^T = R^T Q^T$ certainly is another way of doing this).

13. [M] Investigate the use of QR factorisation in the solution of linear systems of equations: generate a random square matrix, a right hand side vector of ones and the solution using LU factorisation and forward and back substitution using something like

```
A = randn(8, 8); b = ones(8, 1); x = A\b
```

and compare x to the solution y, say using QR factorisation:

```
[Q, R] = qr(A)
y = R\(Q'*b)
```

Note here that \ used with a triangular matrix should perform the appropriate (forwards or backwards) substitution whereas for a general matrix it performs PLU factorisation. You may want to use `format long` to see more digits in x and y since they should be very similar!

14. (*Optional*) **[M]** Using a loop and `tic` and `toc` compare the time it takes to do PLU and QR factorisations. For example, for random matrices of dimension 2^5 to 2^{10}

```
for k=5:10, A=randn(2^k); tic, [L,U,P]=lu(A); toc,...
tic, [Q,R]=qr(A); toc, end
```

should give some timings. (Note how you use dots in MATLAB to continue onto the next line). What do you think the computational work is for QR factorisation given that LU is to leading order $\frac{2}{3}n^3$? Note `qr` uses Householder matrices as described in lectures to compute the QR factorisation.

15. (Optional, specimen exam question for revision)

(a) Define what it means for two n-vectors x and y to be orthogonal, and what it means for an $n \times n$ matrix Q to be orthogonal. (3 marks)

(b) Suppose an $n \times n$ matrix A has LU and QR factorisations

$$A = LU, \qquad A = QR$$

in the usual senses of these terms. State precisely what forms the matrices L, U, Q and R have in these factorisations. (3 marks)

(c) Name (but do not describe) the standard algorithms for computing LU and QR factorisations. (2 marks)

(d) Suppose A is nonsingular. Show that this implies that the diagonal entries of U and of R are all nonzero. (4 marks)

(e) Let $q_1, \ldots, q_n$ denote the columns of Q, and let $\ell_1, \ldots, \ell_n$ denote the columns of L. For any k with $1 \leq k \leq n$, let Q_k and L_k denote the k-dimensional subspaces

$$Q_k = \mathtt{span}\{q_1, \ldots, q_k\}, \qquad L_k = \mathtt{span}\{\ell_1, \ldots, \ell_k\},$$

i.e. the subspaces of all linear combinations of the indicated k vectors. Still assuming A is nonsingular, show that for each k,

$$Q_k = L_k.$$

(7 marks)

(f) Let m denote the last row of L^{-1}. Still assuming A is nonsingular, show that

$$m = Cq_n^T$$

for some constant C. What is this constant? (6 marks)