Part A Numerical Analysis, Hilary 2018. Problem Sheet 6

Note: some questions are marked optional; its up to you and your tutor if you submit these.

1. *(Optional.)* Verify that $\max_{x\in[a,b]} |f(x)|$ defines a norm on $C[a,b]$.

2. For each of the following, say if it defines a norm on $C^1[a,b]$ (the vector space of continuously differentiable functions on $[a,b]$), and if not, why not:

(i) $\left|\int_a^b f(x)\,\mathrm{d}x\right|$

(ii) $\max_{x\in[a,b]} |f(x)+f'(x)|$

(iii) $\max_{x\in[a,b]} [f(x)]^2$

(iv) $\max_{x\in[a,b]} \left\{|f(x)|+|f'(x)|\right\}$

3. *(Optional.)* For vectors $v=(v_1,v_2,\ldots,v_n)^T\in\mathbb{R}^n$ verify that (i) $\|v\|_\infty=\max_{1\le i\le n}|v_i|$ and (ii) $\|v\|_1=\sum_{i=1}^{n}|v_i|$ define norms and draw the 'unit ball' in $\mathbb{R}^2$ (i.e. $\{v\in\mathbb{R}^2:\|v\|\le 1\}$) in each case.

4. **[M]** The MATLAB `norm` command applied to a vector $v\in\mathbb{R}^n$ gives the vector two-norm $\|v\|_2=\left(\sum_{i=1}^{n}v_i^2\right)^{1/2}$ whilst `norm(v,'inf')` gives the value of the norm in Question 3(i) and `norm(v,1)` gives the norm in Question 3(ii). If $v^{(r)}=(1/1,1/2,\ldots,1/r)^T\in\mathbb{R}^r$, experiment to see if $\{\|v^{(r)}\|_k : r=1,2,3,\ldots\}$ is a convergent sequence of real numbers for (a) $k=2$, (b) $k=\inf$, (c) $k=1$. Analytically, what is the answer?

5. *(Optional.)* Is

$$\langle f,g\rangle=\int_a^b f(x)g(x)+f'(x)g'(x)\,\mathrm{d}x$$

an inner product on continuously differentiable functions on $[a,b]$?

6. In the inner product $\langle f,g\rangle=\int_0^2 xf(x)g(x)\,\mathrm{d}x$, calculate the angle between $1/\sqrt{x}$ and $3x$. Can you calculate the angle if the inner product is $\langle f,g\rangle=\int_0^2 f(x)g(x)\,\mathrm{d}x$?

7. Calculate the best approximation to x^3 on $[0,2]$ from Π_2 in the norm derived from the inner product

$$\int_0^2 x f(x) g(x) \,\mathrm{d}x = \langle f, g \rangle.$$

[Hint: MATLAB is very good at solving systems of linear equations as you know by now!]

8. *(Optional.)* Calculate the best approximation to x on $[0,\pi]$ from $V = \operatorname{span}\{1, \cos x\}$ in the norm associated with the inner product

$$\langle f, g \rangle = \int_0^\pi f(x) g(x) \,\mathrm{d}x.$$

(You might want to consider using calculus).

9. By considering $\|f - (p + \epsilon q)\|^2$ where $\epsilon \in \mathbb{R}$, $q \in \Pi_n$, show that if $p \in \Pi_n$ is a best approximation to f in this norm with associated inner product $\langle \cdot, \cdot \rangle$ then $\langle f - p, q \rangle = 0$ for any $q \in \Pi_n$.

10. [**M**] Use MATLAB to solve the problem as in Question 7 but with $p \in \Pi_3$: note that the (i,j)th entry of the coefficient matrix is of the form

$$\frac{2^{i+j+2}}{i+j+2}, \quad i, j = 0, 1, \ldots, n. \tag{1}$$

Comment on your answer! What will be the best approximation if Π_4 is used?

The $(n+1) \times (n+1)$ coefficient matrix A formed when finding the best fit polynomial $p \in \Pi_n$ to a general function f in the norm corresponding to *this* inner product will always have coefficients of the form (1). Calculate the eigenvalues of A for $n = 2, 3, 4, 5$ (I suggest you just use `eig`): you will note that the smallest eigenvalue gets closer to zero and the largest one grows as n is increased—these matrices are becoming more 'ill-conditioned' as n increases, which is the result of choosing the 'bad basis' $\{1, x, x^2, \ldots, x^n\}$ for Π_n rather than an orthogonal one. This ill-conditioning makes the system of equations difficult to solve; how many accurate decimal places are there in the coefficients of the approximation polynomial to $f(x) = x^3$ for $n = 3, 5, 10$?

In this question you may want to use `format long` to increase the number of displayed digits.

11. Specimen Exam question

Given that n is a non-negative integer, let $\mathcal{P}_n$ denote the set of all polynomials in x of degree n of less. Suppose that f is a function, defined and continuous on the interval $[-1, 1]$ of the real line.

(a) What does it mean to say that $p_n \in \mathcal{P}_n$ is the polynomial of best least-squares approximation for f on $[-1, 1]$ with respect to the norm $\|\cdot\|_2$ defined by

$$\|f\|_2 = \left(\int_{-1}^{1} |f(x)|^2 \, \mathrm{d}x\right)^{1/2} ?$$

(3 marks)

(b) Consider the function $f : x \mapsto x^4$ on the interval $[-1, 1]$. By constructing a suitable set of orthogonal polynomials on $[-1, 1]$, or otherwise, find the polynomial $p_3 \in \mathcal{P}_3$ of best least-squares approximation for f on $[-1, 1]$ with respect to the norm $\|\cdot\|_2$, and verify that $p_3(-x) = p_3(x)$ for all $x \in [-1, 1]$. (15 marks)

(c) Now suppose that f is any real-valued function, defined and continuous on the interval $[-1, 1]$ such that $f(-x) = f(x)$ for all $x \in [-1, 1]$. Show that the polynomial $p_n \in \mathcal{P}_n$ of best least-squares approximation for f on the interval $[-1, 1]$ has the property $p_n(-x) = p_n(x)$ for all $x \in [-1, 1]$. (7 marks)