Part A Numerical Analysis, Hilary 2018. Problem Sheet 7

Note: some questions are marked optional; its up to you and your tutor if you submit these.

1. Calculate the orthogonal polynomials ϕ_0, ϕ_1, ϕ_2 in

$$\langle f, g\rangle = \int_0^2 x f(x) g(x)\,\mathrm{d}x,$$

and hence solve again Question 7 from Problem Sheet 6.

2. If $\{\phi_0, \phi_1, \ldots, \phi_n, \ldots\}$ are orthogonal polynomials in $\langle\cdot,\cdot\rangle$ which are normalised to be monic (i.e. have leading coefficient equal to 1) show that $\|\phi_k\| \le \|q\|$ for all monic polynomials $q \in \Pi_k$ which are of exact degree k where $\|\cdot\|$ is the norm derived from the inner product.

3. Let $\mu_j = \int_a^b x^j w(x)\,\mathrm{d}x$ be the jth *moment* of the weight distribution $w(x)$. Show that the linear system of equations

$$\begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \mu_n \\ \mu_{n+1} \\ \vdots \\ \mu_{2n-1} \end{bmatrix}$$

has as solution the coefficients of a polynomial $x^n - \sum_{j=0}^{n-1} c_j x^j$, which is a member of the family of orthogonal polynomials associated with the weight function w.

4. *(Optional.)* If S is a linear spline interpolant of f on equally spaced points in $[a, b]$ what is $\int_a^b S(x)\,\mathrm{d}x$?

5. *(Optional.)* Explicitly construct the cubic spline S which interpolates the data $0, \frac{1}{4}, 1, \frac{1}{4}, 0$ at knots $-2, -1, 0, 1, 2$ respectively, and satisfies $S'(\pm 2) = 0$. [Hint: the first 'piece' is $\frac{1}{4}(x+2)^3$ on $[-2, -1]$ and consider symmetry.]

Verify that S'' is continuous at $x = 0$ and check that $S''(0) = -3$

6. **[M]** Use `spline` in MATLAB to compute the spline in the question above and then plot it (using `ppval` and `linspace`).

7. *(Optional.)* **[M]** Use `spline` in MATLAB to compute the cubic spline interpolant to $\tanh(10x)$ on $[-5, 5]$ with knots $-5, -4, -3, \ldots, 4, 5$ and end conditions $S'(\pm 5) = 0$. Compare with the lagrange interpolant on the same knots.

8. *(Optional.)* By considering the identity

$$\int_{x_0}^{x_n} T''(x)^2 \, \mathrm{d}x - \int_{x_0}^{x_n} S''(x)^2 \, \mathrm{d}x$$

$$= \int_{x_0}^{x_n} \big(T''(x) - S''(x)\big)^2 \, \mathrm{d}x + 2 \int_{x_0}^{x_n} \big(T''(x) - S''(x)\big) \, S''(x) \, \mathrm{d}x$$

prove that amongst all functions $T \in C^2[x_0, x_n]$ which interpolate f at $x_0 < x_1 < \cdots < x_n$, the function which minimises

$$\int_{x_0}^{x_n} T''(x)^2 \, \mathrm{d}x$$

is the natural cubic spline S.

9. *(Optional.)* Calculate the Hermite cubic function which interpolates

x	-2	-1	0	1	2
f	0	$\frac{1}{4}$	1	$\frac{1}{4}$	0
f'	0	α	0	$-\alpha$	0

with $\alpha = 1$. What is special about the interpolating Hermite cubic if $\alpha = \frac{3}{4}$?

10. Show that the set of *natural* cubic splines on a given knot partition $x_0 < x_1 < \cdots < x_n$ is a vector space, V say. Show that V is of dimension $n + 1$.

Why is differentiation *not* a linear transformation from V to V? What is the image of V under the operation of taking the second derivative?

11. *(Optional.)* On a regular partition of knots $0, 1, 2, \ldots, n$ let S be the interpolating natural cubic spline to $f \in C[a, b]$. (Suppose B_0, B_1, B_{n-1} and B_n are defined to satisfy the natural end conditions and consequently satisfy $B_0(1) = \frac{1}{6} = B_n(n-1)$, $B_1(2) = \frac{1}{4}$, $B_1(0) = 0$, $B_{n-1}(n-2) = \frac{1}{4}$, $B_{n-1}(n) = 0$.) If S is expressed as $\sum_{i=0}^{n} \alpha_i B_i(x)$ in terms of the cubic B-splines show that the coefficients (and hence S) can be determined from solution of the linear system

$$\begin{bmatrix} 1 & 0 & & & & \\ \frac{1}{6} & 1 & \frac{1}{4} & & & \\ & \frac{1}{4} & \ddots & \ddots & & \\ & & \ddots & \ddots & \frac{1}{4} & \\ & & & \frac{1}{4} & \ddots & \frac{1}{6} \\ & & & & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ f(n) \end{bmatrix}.$$

By using Gershgorin's theorem (or otherwise) prove that this matrix is non-singular and deduce that the B-splines $B_0, B_1, B_2, \ldots, B_{n-1}, B_n$ defined here form a basis for the vector space of natural cubic splines.

12. Show that one step of Richardson extrapolation applied to the composite Trapezium rule gives the composite Simpson's rule.

13. (Optional, specimen exam question for revision)

Given $n+1$ points $x_0 < x_1 < \cdots < x_n$ and a smooth function f defined on $[x_0, x_n]$, let

$$S_j(x) = \begin{cases} a_j x^3 + b_j x^2 + c_j x + d_j & \text{for } x \in [x_{j-1}, x_j] \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, \ldots, n$ and

$$S(x) = \begin{cases} \sum_{j=1}^{n} S_j(x) & \text{for } x \in (x_0, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_{n-1}, x_n) \\ f(x_i) & \text{for } x = x_i,\ i = 0, 1, \ldots, n. \end{cases}$$

What conditions on the function $S_i(x)$ must be satisfied so that $S(x)$ is a *natural* cubic spline function? (You should assume that these conditions can be satisfied and that the resulting interpolatory natural cubic spline function $S(x)$ is unique.) If $p(x)$ is any linear polynomial, explain why $S(x) + p(x)$ is the interpolatory natural cubic spline for $f(x) + p(x)$. Is this true if $p(x)$ is quadratic?

If $n = 2$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$ and $f(-1) = \alpha$, $f(0) = 0$, $f(1) = \beta$, explicitly determine the coefficients of $S_1(x)$ and $S_2(x)$ in terms of α and β so that $S(x)$ is the interpolatory natural cubic spline for this data. Show that

$$\int_{-1}^{1} S(x)\,\mathrm{d}x = \frac{3}{8}(\alpha + \beta).$$

For an arbitrary smooth function $g(x)$, let $T(x)$ be the interpolatory natural cubic spline function on the knots $-1, 0, 1$. Using the above, show that the approximation to

$$\int_{-1}^{1} g(x)\,\mathrm{d}x$$

obtained by exactly evaluating

$$\int_{-1}^{1} T(x)\,\mathrm{d}x$$

is the quadrature formula

$$\int_{-1}^{1} g(x)\,\mathrm{d}x \approx \int_{-1}^{1} T(x)\,\mathrm{d}x = \frac{3}{8}g(-1) + \frac{5}{4}g(0) + \frac{3}{8}g(1).$$