2 The Green's function

These lecture notes are based on material written by Derek Moulton. *Please send any corrections or comments to* Peter Howell.

2.1 Properties of the Green's function

We recall from §1.7 that the solution of the second-order inhomogeneous ODE

$$\mathfrak{L}y = P_2 y'' + P_1 y' + P_0 y = f \quad a < x < b, \tag{2.1}$$

subject to the simple boundary conditions

$$y(a) = 0 = y(b), \tag{2.2}$$

may be written as

$$y(x) = \int_a^b g(x,\xi) f(\xi)\,\mathrm{d}\xi, \tag{2.3}$$

where the *Green's function* is given by

$$g(x,\xi) = \begin{cases} \dfrac{y_1(\xi)y_2(x)}{P_2(\xi)W(\xi)} & a < \xi < x < b, \\ \dfrac{y_2(\xi)y_1(x)}{P_2(\xi)W(\xi)} & a < x < \xi < b. \end{cases} \tag{2.4}$$

Here y_1 and y_2 are linearly independent solutions of the homogeneous ODE $\mathfrak{L}y = 0$ satisfying one boundary condition each, i.e. $y_1(a) = 0 = y_2(b)$.

We note that the construction of g depends only on the solution of the homogeneous ODE (2.1) and the imposed boundary conditions: it does not depend at all on f. If we are given the linear operator $\mathfrak{L}$ and suitable boundary conditions, in principle we can solve for g "once and for all", and then use (2.3) to give us the solution for arbitrary right-hand side f.Thus the Green's function provides a kind of *inverse* to the differential operator $\mathfrak{L}$ in the sense that $\mathfrak{L}y = f$ (plus suitable boundary conditions) is equivalent to $y = \mathfrak{L}^{-1}f$, with $\mathfrak{L}^{-1}$ defined by (2.3).

It is easily verified that the Green's function defined by (2.4) has the following properties.

(i) $g(x,\xi)$ (viewed as a function of x) satisfies the homogeneous ODE (H) everywhere other than the special point $x = \xi$, i.e.

$$\mathfrak{L}_x g = P_2(x) g_{xx} + P_1(x) g_x + P_0(x) g = 0 \tag{2.5}$$

in $a < x < \xi < b$ and in $a < \xi < x < b$. (Note here for clarity the subscript x indicates that the derivatives are with respect to x rather than ξ.)

(ii) $g(x,\xi)$ (again viewed as a function of x) satisfies the same boundary conditions as y, i.e. $g(a,\xi) = g(b,\xi) = 0$.

(iii) $g(x,\xi)$ is continuous at $x = \xi$, i.e.

$$\lim_{x\to\xi+} g(x,\xi) = \lim_{x\to\xi-} g(x,\xi). \tag{2.6}$$

However, the first derivative of g is discontinuous, with a jump given by

$$\lim_{x\to\xi+} g_x(x,\xi) - \lim_{x\to\xi-} g_x(x,\xi) = \frac{1}{P_2(\xi)}\,. \tag{2.7}$$

2.2 Reverse-engineering g

Suppose we start from the form of the solution (2.3), and try to work out what properties g must have to make (2.3) satisfy the given BVP. Considering first the boundary conditions (2.2), we get

$$\int_a^b g(a,\xi)f(\xi)\,\mathrm{d}\xi = 0 = \int_a^b g(b,\xi)f(\xi)\,\mathrm{d}\xi \quad \text{for all functions } f(\xi), \tag{2.8}$$

which indeed leads us to property (ii) above.

Second, let us substitute (2.3) into the ODE (2.1), assuming (a risky assumption as we will see) that the x-derivatives may be passed through the integral sign so that

$$\mathfrak{L}\int_a^b g(x,\xi)f(\xi)\,\mathrm{d}\xi = \int_a^b \mathfrak{L}_x g(x,\xi)f(\xi)\,\mathrm{d}\xi = f(x). \tag{2.9}$$

To make this work, we apparently need g to satisfy

$$\mathfrak{L}_x g(x,\xi) = \delta(x-\xi), \tag{2.10}$$

where δ is a function (if one exist) with the property that

$$\int_a^b \delta(x-\xi)\phi(\xi)\,\mathrm{d}\xi \equiv \phi(x), \tag{2.11}$$

for any (suitably smooth) function ϕ. The property (2.11) is known as the *sifting property* — δ is somehow supposed to pick out the value of the test function ϕ at a specific point. Luckily, a function with the property (2.11) does exist (though it isn't really a function) and is called the Dirac delta function.

2.3 The delta function

2.3.1 Definition

The delta function may be thought of as describing a point source, and may be characterised by the properties

$$\delta(x) = 0 \quad \text{for all } x \neq 0, \tag{2.12a}$$

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1. \tag{2.12b}$$

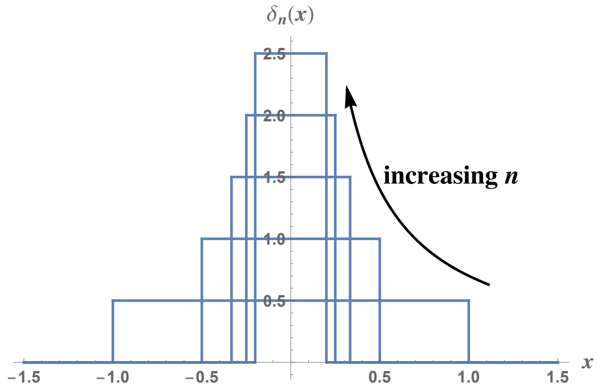

Figure 2.1: Hat functions defined by equation (2.16).

The first property (2.12a) captures the notion of a point function. The second property (2.12a) constrains the area under the graph (which you might think of as infinitely thin and infinitely high). An idealized unit point source at $x = 0$ is described by $\delta(x)$; a point source at some other point $x = \xi$ would be given by $\delta(x - \xi)$.

If a δ existed satisfying (2.12), then it would also have the desired sifting property (2.11). By property (2.12), for any $x \in (a, b)$ we can write

$$\int_a^b \delta(x-\xi)\phi(\xi)\,\mathrm{d}\xi = \int_{x-\epsilon}^{x+\epsilon} \delta(x-\xi)\phi(\xi)\,\mathrm{d}\xi, \tag{2.13}$$

where ϵ is an arbitrarily small positive parameter. For sufficiently smooth ϕ, we can thus approximate

$$\int_a^b \delta(x-\xi)\phi(\xi)\,\mathrm{d}\xi \sim \big[\phi(x) + O(\epsilon)\big] \int_{x-\epsilon}^{x+\epsilon} \delta(x-\xi)\,\mathrm{d}\xi, \tag{2.14}$$

and by letting $\epsilon \to 0$, we find that the right-hand side is equal to $\phi(x)$ as required.

2.3.2 Approximating the delta function

The problem is that no classical function satisfies both properties (2.12) (any function that is non-zero only at a point either is not integrable or integrates to zero). One way around this difficulty is to replace δ by an approximating sequence of increasingly narrow functions with normalised area, i.e. $\delta_n(x)$ where

$$\int_{-\infty}^{\infty} \delta_n(x)\,\mathrm{d}x = 1 \quad \text{for all } n = 1,\, 2,\, \ldots, \tag{2.15a}$$

$$\lim_{n\to\infty} \delta_n(x) = 0 \quad \text{for all } x \neq 0. \tag{2.15b}$$

One possibility is “hat” functions of the form

$$\delta_n(x) = \begin{cases} 0 & \text{for } |x| > 1/n, \\ n/2 & \text{for } |x| \leq 1/n. \end{cases} \tag{2.16}$$

It is easily verified that the sequence of functions $\delta_n(x)$ defined by (2.16) has the desired properties (2.15). As illustrated in figure 2.1, as n increases, $\delta_n(x)$ approaches a “spike”, equal to zero everywhere apart from a neighbourhood of the origin but nevertheless with unit area under the graph.

2.3.3 Properties of delta function

Approximating sequences like (2.16) can be used to establish various properties of the delta function.

Sifting property Let $\phi(x)$ be a smooth function, and $\Phi(x) = \int \phi(x)\,\mathrm{d}x$ its antiderivative. If we use the particular approximating sequence (2.16), then

$$\int_{-\infty}^{\infty} \delta_n(x-a)\phi(x)\,\mathrm{d}x = \int_{a-1/n}^{a+1/n} (n/2)\phi(x)\,\mathrm{d}x = \frac{n}{2}\left[\Phi(a+n/2) - \Phi(a-n/2)\right]. \tag{2.17}$$

Now letting $n \to \infty$ we get

$$\int_{-\infty}^{\infty} \delta_n(x-a)\phi(x)\,\mathrm{d}x \to \Phi'(a) = \phi(a). \tag{2.18}$$

Therefore δ does have the desired sifting property

$$\int_{-\infty}^{\infty} \delta(x-a)\phi(x)\,\mathrm{d}x \equiv \phi(a) \tag{2.19}$$

(for suitably smooth test functions ϕ) *if* we make the identification that

$$\int_{-\infty}^{\infty} \delta(x-a)\phi(x)\,\mathrm{d}x \equiv \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x-a)\phi(x)\,\mathrm{d}x. \tag{2.20}$$

This final identification (2.20) is *not valid* in the space of classical functions (the convergence of δ_n to δ is non-uniform) but it *does* hold for so-called *distributions*. Rather than trying to approximate δ with a classical function, instead, one defines it as a linear functional on the space of "test functions" $\mathcal{T}$:

$$\delta : \mathcal{T} \to \mathbb{R}, \tag{2.21a}$$

$$\delta : \phi(x) \mapsto \phi(0). \tag{2.21b}$$

See ASO Integral Transforms for more details about this more systematic approach.

Antiderivative of δ The antiderivative of the delta function is the so-called *Heaviside function*:

$$\int_{-\infty}^{x} \delta(s)\,\mathrm{d}s = H(x) := \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases} \tag{2.22}$$

(The value of $H(x)$ *at* $x = 0$ is indeterminate: it is sometimes taken to be 1 and sometimes taken to be 1/2.)

Note that (2.22) may be obtained by integrating the sequence (2.16) of approximating functions and showing that the limit is the Heaviside function, that is [**Exercise**]

$$\lim_{n\to\infty} \int_{-\infty}^{x} \delta_n(s)ds = H(x) \tag{2.23}$$

(with the same caveat as above about the validity of taking the limit through the integral). Alternatively, one can convince oneself that $H'(x) = 0$ for $x \neq 0$ but

$$\int_{-\infty}^{\infty} H'(x)\,\mathrm{d}x = \int_{-\epsilon}^{\epsilon} H'(x)\,\mathrm{d}x = \big[H\big]_{-\epsilon}^{\epsilon} = 1, \tag{2.24}$$

for any $\epsilon > 0$. Thus H' has the defining properties (2.12) of δ. Again, all of these arguments can be made more watertight using the theory of distributions.

2.4 Green's function via delta function

Now let us return to the problem of finding a Green's function $g(x, \xi)$ satisfying (2.10). We start by doing a very simple case with $\mathfrak{L}y = y''$ and $y(0) = 0 = y(1)$.

Example 2.1. Find $g(x, \xi)$ satisfying

$$g_{xx}(x, \xi) = \delta(x - \xi) \quad \text{for } 0 < x,\, \xi < 1, \tag{2.25a}$$

$$g(0, \xi) = 0 = g(1, \xi). \tag{2.25b}$$

Since its right-hand side is zero for $x \neq \xi$, we can easily integrate (2.25a) *to obtain the solution in $x < \xi$ and in $x > \xi$. By applying the boundary conditions* (2.25b) *we deduce that*

$$g(x, \xi) = \begin{cases} A(\xi)x & 0 < x < \xi < 1, \\ B(\xi)(1 - x) & 0 < \xi < x < 1, \end{cases} \tag{2.26}$$

where A and B are two arbitrary functions of integration. To evaluate A and B we need to decide how to join the two solutions together across $x = \xi$. To do this, we integrate (2.25a) *across the singularity at $x = \xi$, that is,*

$$\int_{\xi-}^{\xi+} g_{xx}(x, \xi)\, \mathrm{d}x = \int_{\xi-}^{\xi+} \delta(x - \xi)\, \mathrm{d}x$$

$$\Rightarrow \quad \big[g_x(x, \xi)\big]_{x=\xi_-}^{x=\xi_+} = 1, \tag{2.27}$$

where ξ_- and ξ_+ refer to the limits as x approaches ξ from below and from above, respectively. From (2.27) *we deduce that* there must be a unit jump in the derivative of g across the point $x = \xi$. *The second condition to determine A and B is that g* itself must be continuous *(more about this below).*

So we impose the jump conditions

$$\big[g(x, \xi)\big]_{x=\xi_-}^{x=\xi_+} = 0, \qquad \big[g_x(x, \xi)\big]_{x=\xi_-}^{x=\xi_+} = 1, \tag{2.28}$$

to obtain

$$B(1 - \xi) - A\xi = 0, \tag{2.29a}$$

$$-B - A = 1, \tag{2.29b}$$

and hence $A(\xi) = -(1 - \xi)$, $B(\xi) = -\xi$, and the Green's function in this case is given by

$$g(x, \xi) = \begin{cases} -x(1 - \xi) & 0 < x < \xi < 1, \\ -(1 - x)\xi & 0 < \xi < x < 1. \end{cases} \tag{2.30}$$

The Green's function given by (2.30) is sketched in Figure 2.2(a). As we imposed, g satisfies the boundary conditions $g = 0$ at $x = 0$ and $x = 1$, and is continuous everywhere. The first x-derivative of g undergoes a unit jump across $x = \xi$, as shown in Figure 2.2(b), and in fact resembles a Heaviside function. It follows that the second derivative has a delta function at $x = \xi$, as illustrated in Figure 2.2(c).

Now: what would have happened if we didn't impose continuity of g across $x = \xi$? In that case g_x would have a delta function at $x = \xi$ and g_{xx} would have an even worse singularity ($\delta'(x)$, the derivative of the delta function, which is a well-defined distribution). So, continuity of g ensures that we only have a delta-function singularity at $x = \xi$ and nothing stronger.

We illustrate the approach more generally with a less trivial example.

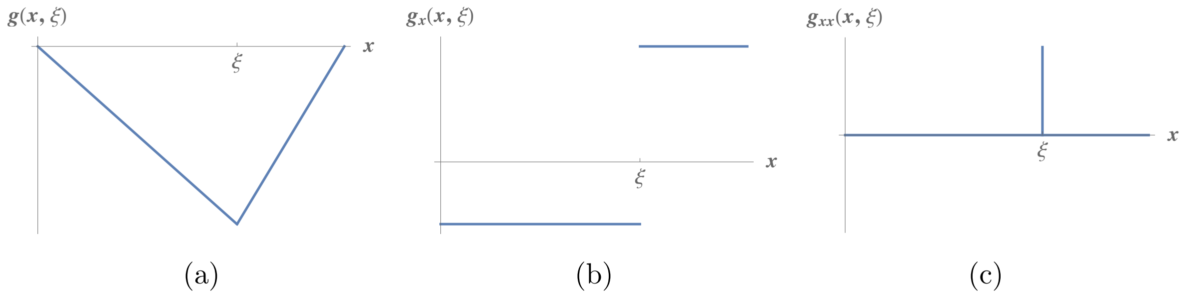

Figure 2.2: The Green's function defined by (2.30) and its first two derivatives (with $\xi = 0.65$).

Example 2.2. Find the Green's function for the problem

$$y''(x) + y(x) = f(x) \quad \text{for } 0 < x < \frac{\pi}{2}, \qquad y(0) = 0 = y\left(\frac{\pi}{2}\right). \tag{2.31}$$

So we have to solve

$$g_{xx} + g = \delta(x-\xi) \quad \textit{for } 0 < x,\, \xi < \frac{\pi}{2}, \qquad g(0,\xi) = 0 = g\left(\pi/2, \xi\right). \tag{2.32}$$

Since the right-hand side is zero for $x \neq \xi$, we can find the solution on either side of the singularity and thus, applying the boundary conditions, we get

$$g(x,\xi) = \begin{cases} A(\xi)\sin x & 0 < x < \xi < 1, \\ B(\xi)\cos x & 0 < \xi < x < 1. \end{cases} \tag{2.33}$$

To derive the appropriate jump conditions, we again integrate (2.32) *across $x = \xi$, as follows:*

$$\int_{\xi-}^{\xi+} g_{xx}(x,\xi) + g(x,\xi)\,\mathrm{d}x = \int_{\xi-}^{\xi+} \delta(x-\xi)\,\mathrm{d}x$$
$$\Rightarrow \quad \big[g_x(x,\xi)\big]_{x=\xi_-}^{x=\xi_+} = 1, \tag{2.34}$$

since the integral of g over the infinitesimal interval $[\xi_-, \xi_+]$ is zero. Again, we impose continuity of g itself (to eliminate any stronger singularity than δ) and thus we have exactly the same jump conditions (2.28) *as above.*

We can then easily solve for A and B and hence obtain

$$g(x,\xi) = \begin{cases} -\cos\xi \sin x & 0 < x < \xi < \frac{\pi}{2}, \\ -\sin\xi \cos x & 0 < \xi < x < \frac{\pi}{2}, \end{cases} \tag{2.35}$$

which agrees exactly with the solution found in Example 1.3 *using variation of parameters.*

We can generalise the above arguments to obtain the appropriate jump conditions for a general second-order linear operator of the form

$$\mathfrak{L}_x g(x,\xi) = P_2(x)g_{xx}(x,\xi) + P_1(x)g_x(x,\xi) + P_0(x)g(x,\xi) = \delta(x-\xi), \tag{2.36}$$

namely

$$\big[g(x,\xi)\big]_{x=\xi_-}^{x=\xi_+} = 0, \qquad \big[g_x(x,\xi)\big]_{x=\xi_-}^{x=\xi_+} = \frac{1}{P_2(\xi)}. \tag{2.37}$$

These conditions reproduce property (iii) of g noted in §2.4. Note once again the importance of P_2 being nonzero on the interval of interest.

Exercise: (i) derive (2.37); (ii) hence obtain the general formula (2.4) for g.

2.5 Generalisation

We now show how to generalise the concepts developed above to linear ODEs of arbitrary order and with more complicated (but still linear) boundary conditions. A general linear differential operator of order $n \in \mathbb{N}$ may be written as

$$\mathfrak{L}y(x) \equiv P_n(x)y^{(n)}(x) + P_{n-1}(x)y^{(n-1)}(x) + \cdots + P_1(x)y'(x) + P_0(x)y(x) \tag{2.38a}$$

$$\equiv P_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + P_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \cdots + P_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + P_0(x)y(x), \tag{2.38b}$$

for some given coefficients $P_0, \ldots, P_n$; (2.38) is equivalent to (1.2) when $n = 2$. As in Section 1, we assume that all P_i are at least continuous and that the coefficient P_n of the highest derivative is nonzero.

In terms of $\mathfrak{L}$, we define homogeneous and inhomogeneous linear ODEs of order n by

$$\mathfrak{L}y = 0, \tag{H}$$

$$\mathfrak{L}y = f \not\equiv 0. \tag{N}$$

In a general nth-order linear BVP, the ODE (N) is supplemented by n boundary conditions, each of which consists of a linear combination of y and its derivatives up to order $n-1$, evaluated at the boundary points $x = a$ and $x = b$. We will write these generically as

$$\mathfrak{B}_i y\Big|_{x=a,b} = \gamma_i, \quad i = 1, 2, \ldots, n, \tag{BCN}$$

where γ_i are constants and each $\mathfrak{B}_i$ is of the form

$$\mathfrak{B}_i y = \sum_{j=1}^{n}\left(\alpha_{ij}y^{(j-1)}(a) + \beta_{ij}y^{(j-1)}(b)\right), \tag{2.39}$$

for some constants α_{ij}, β_{ij} (which must be such that (BCN) comprises n independent equations). For instance, for a 2nd order system, with $n = 2$ the most general linear boundary conditions would have the form

$$\mathfrak{B}_1 y = \gamma_1, \qquad \mathfrak{B}_2 y = \gamma_2, \tag{2.40}$$

where

$$\mathfrak{B}_1 y = \alpha_{11}y(a) + \alpha_{12}y'(a) + \beta_{11}y(b) + \beta_{12}y'(b), \tag{2.41a}$$

$$\mathfrak{B}_2 y = \alpha_{21}y(a) + \alpha_{22}y'(a) + \beta_{21}y(b) + \beta_{22}y'(b), \tag{2.41b}$$

which is equivalent to (BC) in the homogeneous case where $\gamma_1 = \gamma_2 = 0$.

The boundary conditions (BCN) are *homogeneous* if $\gamma_i = 0$ for all i, in which case we have

$$\mathfrak{B}_i y\Big|_{x=a,b} = 0, \quad i = 1, 2, \ldots, n, \tag{BCH}$$

We assume that

the homogeneous problem (H + BCH) *has no non-trivial solutions*, $(\star)$

and then by FAT (Theorem 1.2), we expect the inhomogeneous problem to have a unique solution.

We can reduce the full problem (N+BCN) to one with homogeneous boundary conditions by subtracting off a suitable solution of the homogeneous problem (H). Let u be the solution of the problem (H + BCN), i.e.

$$\mathfrak{L}u(x) = 0, \quad a < x < b, \tag{2.42a}$$

$$B_i u\Big|_{x=a,b} = \gamma_i, \quad i = 1, 2, \ldots, n. \tag{2.42b}$$

It may be shown that, under the assumption ($\star$), a unique solution for u exists. Then defining $\tilde{y} = y - u$, we see that $\tilde{y}$ satisfies the inhomogeneous ODE (N) but with the homogeneous boundary conditions (BCH). We may therefore focus on the problem (N+BCH).

2.6 Green's function for a general BVP

As above, we assume that the boundary conditions have been made homogeneous so we can consider a general nth-order BVP of the form (N+BCH), i.e.

$$\mathfrak{L}y(x) = \sum_{j=1}^{n} P_i(x) y^{(j-1)}(x) = f(x) \quad a < x < b, \tag{2.43a}$$

subject to n linearly independent homogeneous boundary conditions

$$\mathfrak{B}_i y\big|_{x=a,b} = \sum_{j=1}^{n} \Big(\alpha_{ij} y^{(j-1)}(a) + \beta_{ij} y^{(j-1)}(b) \Big) = 0, \quad i = 1, 2, \ldots, n. \tag{2.43b}$$

The corresponding problem for g is

$$\mathfrak{L}_x g(x, \xi) = \delta(x - \xi) \quad a < x, \xi < b, \tag{2.44a}$$

with boundary conditions

$$\mathfrak{B}_i g(x, \xi)\big|_{x=a,b} = 0, \quad i = 1, 2, \ldots, n. \tag{2.44b}$$

Since $\delta(x - \xi)$ is zero for $x \neq \xi$, we can in principle solve (2.44a) to get two distinct solutions in each of the sub-intervals $a < x < \xi$ and $a < \xi < x < b$. Given that $\mathfrak{L}$ is of order n, we will then have $2n$ degrees of freedom, i.e. n arbitrary integration constants. After applying the n independent boundary conditions (2.44b) we will have n remaining constants (actually functions of ξ) to determine. We therefore need n jump conditions at $x = \xi$, which come as above by integrating across $x = \xi$:

$$\int_{\xi-}^{\xi+} \left[P_n(x) \frac{\partial^n}{\partial x^n} g(x, \xi) + \cdots + P_0(x) g(x, \xi) \right] \mathrm{d}\xi = \int_{\xi-}^{\xi+} \delta(x - \xi)\, \mathrm{d}\xi = 1. \tag{2.45}$$

By integrating the first term on the left-hand side by parts, we obtain

$$\int_{\xi-}^{\xi+} \left[\big(P_{n-1}(x) - P_n'(x)\big) \frac{\partial^{n-1}}{\partial x^{n-1}} g(x, \xi) + \cdots + P_0(x) g(x, \xi) \right] \mathrm{d}\xi + \left[P_n(x) \frac{\partial^{n-1}}{\partial x^{n-1}} g(x, \xi) \right]_{x=\xi-}^{x=\xi+} = 1. \tag{2.46}$$

This equation is balanced by setting a jump condition on the $(n-1)$th derivative:

$$\left[\frac{\partial^{n-1}}{\partial x^{n-1}} g(x,\xi)\right]_{x=\xi-}^{x=\xi+} = 1/P_n(\xi), \tag{2.47}$$

and taking all lower derivatives to be continuous across $x = \xi$:

$$\big[g(x,\xi)\big]_{x=\xi-}^{x=\xi+} = \big[g_x(x,\xi)\big]_{x=\xi-}^{x=\xi+} = \cdots = \left[\frac{\partial^{n-2}}{\partial x^{n-2}} g(x,\xi)\right]_{x=\xi-}^{x=\xi+} = 0. \tag{2.48}$$

Once the Green's function is determined, following the above procedure, the solution to the BVP (2.43) is given by

$$y(x) = \int_a^b g(x,\xi) f(\xi)\,\mathrm{d}\xi. \tag{2.49}$$

It can be verified by direct substitution that (2.49) satisfies (2.44), provided (i) g satisfies (2.44) and (ii) it is legitimate to pass the differential operator $\mathfrak{L}$ through the integral in (2.49).

2.7 Green's function in terms of adjoint

There is an alternative way to construct the Green's function that eliminates the need for any dicey differentiating through integrals. Start from the ODE (2.43) and take an inner product with an *a priori* unknown function $G(x,\xi)$ on both sides of the equation to obtain

$$\langle \mathfrak{L}y, G\rangle = \langle G(x,\xi), f(x)\rangle = \int_a^b G(x,\xi) f(x)\,\mathrm{d}x. \tag{2.50}$$

(Note here the integration is with respect to x). Now, using the adjoint, we can write

$$\langle \mathfrak{L}y, G\rangle = \langle y, \mathfrak{L}^* G\rangle = \int_a^b y(x) \mathfrak{L}_x^* G(x,\xi)\,\mathrm{d}x, \tag{2.51}$$

provided G satisfies the *adjoint* boundary conditions corresponding to the boundary conditions (2.43b) imposed on y.

The idea now is to isolate y. This can be accomplished if G satisfies

$$L_x^* G(x,\xi) = \delta(x-\xi) \tag{2.52}$$

(as well as the corresponding adjoint boundary conditions), in which case the right-hand side of (2.51) leaves just $y(\xi)$, and we have the solution

$$y(\xi) = \int_a^b G(x,\xi) f(x)\,\mathrm{d}x. \tag{2.53}$$

To make comparison with our previous construction easier, we can switch the roles of x and ξ to write (2.53) in the equivalent form

$$y(x) = \int_a^b G(\xi,x) f(\xi)\,\mathrm{d}\xi = \int_a^b g(x,\xi) f(\xi)\,\mathrm{d}\xi. \tag{2.54}$$

We deduce that $G(\xi,x) \equiv g(x,\xi)$: we might say that G is the *transpose* of g (*cf* §1.8). In summary, if

- $g(x,\xi)$ satisfies $\mathfrak{L}_x g(x,\xi) = \delta(x-\xi)$ with homogeneous boundary conditions (BC),

then

- $g(\xi,x)$ satisfies the corresponding adjoint equation $\mathfrak{L}_x^* g(\xi,x) = \delta(x-\xi)$ and boundary conditions (BC*).

In particular,

- if ($\mathfrak{L}$ + BC) is fully self-adjoint, then g is *symmetric*, i.e. $g(x,\xi) \equiv g(\xi,x)$ (and *vice versa*).

2.8 FAT and Green's function

As we have seen, the Green's function approach apparently gives the explicit constructive solution to $\mathfrak{L}y = f$ with homogeneous boundary conditions (BC). So, if the Green's function approach works, i.e. if we can find g, then we have both existence and uniqueness of the solution y. But we know from FAT (Theorem 1.2) that, when there is a non-trivial solution of the homogeneous problem ($\mathfrak{L}y = 0$+BC), the solution of the inhomogeneous problem should either not exist or not be unique. Clearly, in such cases something must go wrong with the construction of the Green's function. So, let us suppose that ($\mathfrak{L}y = 0$+BC) admits non-zero solutions and thus, similarly, the adjoint problem ($\mathfrak{L}^* w = 0$+BC*) admits a non-zero solution w. Then, starting from the delta function formulation

$$\mathfrak{L}_x g(x,\xi) = \delta(x-\xi), \tag{2.55}$$

and taking the inner product with w, we get the solvability condition

$$0 = \langle G(x,\xi), \mathfrak{L}^* w(x)\rangle = \langle \mathfrak{L}_x G(x,\xi), w(x)\rangle = \langle \delta(x-\xi), w(x)\rangle = w(\xi) \tag{2.56}$$

which clearly does not hold since $w \not\equiv 0$ (by assumption).

Thus, in situations where ($\mathfrak{L}$+BC) has a non-trivial kernel, we can't construct the Green's function. (One can instead construct a so-called *modified Green's function*, and thus obtain the non-unique form of the solution in case 2(a) of FAT, but we won't go into details here).