4 Power series solution of linear ODEs

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4.1 Singular points of ODEs

4.1.1 Introduction

This section concerns nth order homogeneous linear ODEs of the form

$$\mathfrak{L}y(x) = y^{(n)}(x) + P_{n-1}(x)y^{(n-1)}(x) + \cdots + P_1(x)y^{(n)}(x) + P_0(x)y(x) = 0. \tag{4.1}$$

Note, in comparison with (2.38), we have divided through by $P_n(x)$ so that the coefficient of the highest-order derivative $y^{(n)}(x)$ is equal to 1. We will seek the solution to (4.1) in the form of a *power series expansion* in the neighbourhood of some point $x = x_0$. Both the procedure and the nature of the solution depend on how well-behaved the functions $P_j(x)$ are as $x \to x_0$.

4.1.2 Ordinary points

The point x_0 is an *ordinary point* of the ODE (4.1) if all $P_j(x)$ are *analytic* in a neighbourhood of $x = x_0$, i.e. they each have a convergent power series expansion of the form $\sum_{k=0}^{\infty} c_k(x-x_0)^k$. In this case, it may be shown that:

1. all n linearly independent solutions of (4.1) are also analytic in a neighbourhood of $x = x_0$, i.e. can be expressed in the form
$$y(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k; \tag{4.2}$$
2. the radius of convergence of the series solution (4.2) $\geq$ distance (in $\mathbb{C}$) to nearest singular point of the coefficent functions $P_j(x)$.

The procedure at an ordinary point is straightforward: just (i) plug the expansion (4.2) into the ODE (4.1), using the power series expansions of each of the P_j, then (ii) by equating the coefficient of each power of x to zero, obtain a sequence of equations for the coefficients a_k that can be solved recursively.

Example 4.1. Find the solution of

$$y'(x) + \frac{2x}{(1+x^2)}\,y(x) = 0 \tag{4.3}$$

as a power series expansion about $x = 0$.

Here $x_0 = 0$ is an ordinary point. The nearest singular points of $P_0(x) = 2x/\left(1+x^2\right)$ are at $x = \pm i$, distance 1 from 0, so the solution of (4.3) can be written as a regular power series expansion whose radius of convergence $R \geq 1$.

By substituting (4.2) into (4.3) and multiplying through by $\left(1+x^2\right)$, we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} \left[\left(1+x^2\right) k a_k x^{k-1} + 2a_k x^{k+1}\right] \\ &= \sum_{k=0}^{\infty} \left[k a_k x^{k-1} + (k+2) a_k x^{k+1}\right]. \end{aligned} \tag{4.4}$$

Now we want to increase k by 2 in the first term in the sum so that the exponents of x agree: we have to take care of the cases $k = 0$ and $k = 1$ separately and so end up with

$$0 = 0 \times a_0 x^{-1} + 1 \times a_1 + \sum_{k=0}^{\infty} \left[(k+2) a_{k+2} x^{k+1} + (k+2) a_k x^{k+1}\right]. \tag{4.5}$$

The coefficient of x^{-1} is zero identically. By setting the coefficient of x^0 to zero, we deduce that a_1 must be zero. Then by setting to zero all the coefficients of x, x^2, x^3, ..., we get the recurrence relation

$$a_{k+2} = -a_k \quad (k = 0,\, 1,\, 2,\, \ldots). \tag{4.6}$$

Since $a_1 = 0$, it follows that the odd coefficients a_3, a_5, ... are all equal to zero, and the even coefficients are given by $a_{2k} = (-1)^k a_0$. The solution of (4.3) is thus given by

$$y(x) = a_0 \sum_{k=0}^{\infty} (-1)^k x^{2x}. \tag{4.7}$$

One can easily verify that the radius of convergence of the series (4.7) is equal to 1. Indeed, it is easy to solve the simple ODE (4.3) exactly to get $y(x) = \text{const}/\left(1+x^2\right)$, of which (4.7) is just the Maclaurin expansion.

4.1.3 Singular points

The point x_0 is called a *singular point* of the ODE (4.1) if at least one of the coefficient functions $P_j(x)$ is not analytic there. In this case, the general solution $y(x)$ may not be analytic at x_0: $y(x)$ or its derivatives might "blow-up" as $x \to x_0$. The following simple example illustrates how solutions can behave near a singular point.

Example 4.2. *Consider the first-order ODE*

$$y'(x) - \lambda x^{-m} y(x) = 0, \tag{4.8}$$

where $\lambda \in \mathbb{R}$ and m is a non-negative integer. The general solution of (4.8) can easily be found via separation of variables, and the generic behaviour as $x \to 0$ depends on the value of m.

(i) *For $m = 0$, the point $x = 0$ is ordinary. The solution $y(x) = \text{const} \times e^{\lambda x}$ can be expanded as a power series about $x = 0$ which converges for all $x \in \mathbb{C}$.*

(ii) *For $m = 1$, the point $x = 0$ is singular. The solution in this case is $y(x) = \text{const} \times x^{\lambda}$, which is analytic if λ is a non-negative integer (despite 0 being a singular point). For any other λ, the solution is singular at $x = 0$, but with a relatively benign singularity: either a pole (if λ is a negative integer) or a branch point (otherwise).*

(iii) *For $m = 2$, the behaviour is much worse: the solution of (4.8) is $y(x) = \text{const} \times \exp(-\lambda/x)$, which has an* essential singularity *at $x = 0$. Similarly, there is an essential singularity at $x = 0$ for any value of $m \geq 2$.*

Example 4.2 suggests that the strength of the singularity in the solution at a singular point tends to increase the higher the order of the poles in the coefficients in front of the lower order terms of the ODE. Indeed, this is the key idea behind the classification of singular points.

4.1.4 Regular singular points

If the coefficients $P_j(x)$ are not all analytic at $x = x_0$, but the modified coefficients

$$p_j(x) \equiv P_j(x)(x - x_0)^{n-j} \text{ are all analytic at } x = x_0, \tag{4.9}$$

then $x = x_0$ is a *regular singular point* of the ODE (4.1). For example, Case (ii) of Example 4.2 has a regular singular point at $x = 0$. For the general second-order ODE

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \tag{4.10}$$

there is a regular singular point at $x = x_0$ if at least one of $P(x)$ and $Q(x)$ is not analytic at $x = x_0$ but both $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are.

Any singular point that does not satisfy the criterion (4.9) is an *irregular singular points.* At a regular singular point, the singularity in the solution is "not too bad", and a modification of the power series approach can be used. For irregular singular points, though, there is no general theory!

Example 4.3. Cauchy–Euler equation

The Cauchy–Euler equation

$$y''(x) + \frac{a}{x}\, y(x) + \frac{b}{x^2}\, y(x) = 0 \tag{4.11}$$

has a regular singular point at $x = 0$. The general solution can be found via the ansatz $y = x^{\alpha}$, where α satisfies the characteristic equation $\alpha(\alpha - 1) + a\alpha + b = 0$, and there are two cases to consider.

(i) *If the characteristic equation has two distinct roots α_1 and α_2, then, the general solution of* (4.11) *is given by*

$$y(x) = C_1 x^{\alpha_1} + C_2 x^{\alpha_2} \tag{4.12}$$

(where C_1 and C_2 are arbitrary constants).

(ii) *If the characteristic equation has a double root α, then the general solution is*

$$y(x) = C_1 x^{\alpha} + C_2 x^{\alpha} \log x. \tag{4.13}$$

Note that if the roots are two distinct non-negative integers, then the general solution in case (i) *is analytic (even though the ODE has a singular point). In general, however, the behaviour as $x \to 0$ could be a negative, fractional or even complex power of x, and the solution generically has a pole or a branch point at $x = 0$.*

The behaviour illustrated by Example 4.3 carries over to regular singular points in general, except that the functions x^{α_1} and x^{α_2} are each multiplied by an analytic function (i.e. a regular power series in x). The general theory for regular singular points will be explained below, but first we show how the point at infinity can be analysed.

4.1.5 The point at infinity

The point $x_0 = \infty$ can also be classified by changing the independent variable via the substitution

$$t = 1/x, \qquad u(t) = y(x), \tag{4.14}$$

and classifying the point $t = 0$ for the resulting ODE for $u(t)$.

Example 4.4. Find and classify the singular points of the ODEs

(i) $y'(x) - y(x) = 0,$

(ii) $y''(x) + \dfrac{1}{x}\, y(x) = 0.$

In case (i)*, the coefficient* $P_0(x) = -1$ *is analytic everywhere, and there don't appear to be any singular points. But if we make the change of variables* (4.14) *then, by the chain rule, we have* $u'(t) = -\left(1/t^2\right) y'(x)$*. The ODE* (i) *therefore becomes*

$$u'(t) + \frac{1}{t^2}\, u(t) = 0, \tag{4.15}$$

which has an irregular singular point *at* $t = 0$*, and it follows that* (i) *has an irregular singular point at* $x = \infty$*. (Indeed, the solution* $y(x) = \mathrm{e}^x$ *has an essential singularity as* $x \to \infty$*.)*

In case (ii)*, there is a regular singular point at* $x = 0$ *(since* $x^2 \times (1/x) = x$ *is analytic at* $x = 0$*). Again making the substitution* (4.14)*, we get* [**exercise**]

$$u''(t) + \frac{2}{t}\, u'(t) + \frac{1}{t}\, u(t) = 0, \tag{4.16}$$

which likewise has a regular singular point at $t = 0$*. Therefore the ODE* (ii) *has regular singular points at* $x = 0$ *and at* $x = \infty$*.*

4.2 Frobenius method for 2nd order ODEs

4.2.1 The indicial equation

From now on, we restrict attention to regular singular points of 2nd order ODEs. If $x = x_0$ is a regular singular point, then we can write the ODE in the form

$$\mathfrak{L}y(x) = y''(x) + \frac{p(x)}{(x - x_0)}\, y'(x) + \frac{q(x)}{(x - x_0)^2}\, y(x) = 0, \tag{4.17}$$

where p and q are analytic, and can therefore be expanded as convergent power series:

$$p(x) = \sum_{k=0}^{\infty} p_k (x - x_0)^k, \qquad q(x) = \sum_{k=0}^{\infty} q_k (x - x_0)^k. \tag{4.18}$$

The idea is to seek a solution in the form of a *Frobenius series*

$$y(x) = (x - x_0)^{\alpha} \sum_{k=0}^{\infty} a_k (x - x_0)^k. \tag{4.19}$$

Note the similarity to the Cauchy–Euler Example 4.3: $y(x) \sim a_0 (x - x_0)^{\alpha}$ as $x \to 0$, but now the power of x is multiplied by an *a priori* unknown analytic function $\sum_k a_k (x - x_0)^k$, with coefficients a_k to be determined. We may assume that $a_0 \neq 0$ by choosing α appropriately.

Now we plug (4.19) into the ODE (4.17), to get

$$\sum_{k=0}^{\infty}(\alpha+k)(\alpha+k-1)a_k(x-x_0)^{\alpha+k-2}+\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(\alpha+k)p_j a_k(x-x_0)^{\alpha+k+j-2} \\ +\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}q_j a_k(x-x_0)^{\alpha+k+j-2}=0, \quad (4.20)$$

and equate coefficients. At the lowest power, namely $(x-x_0)^{\alpha-2}$, we find

$$\big[\alpha(\alpha-1)+p_0\alpha+q_0\big]a_0=0. \quad (4.21)$$

Since a_0 is defined to be non-zero, the quadratic function in brackets must be zero. This polynomial plays an important role, and we will denote it by

$$F(\alpha)=\alpha(\alpha-1)+p_0\alpha+q_0. \quad (4.22)$$

The equation $F(\alpha)=0$ is called the *indicial equation*, and it determines the possible indicial exponents α_1, α_2. Note that in general these exponents can be complex! In any case, we order them such that $\mathrm{Re}[\alpha_1]\geq\mathrm{Re}[\alpha_2]$.

4.2.2 The first series solution

Let us continue equating coefficients of powers of $(x-x_0)$. We find after some algebra that the coefficients of $(x-x_0)^{k+\alpha-2}$ satisfy

$$F(\alpha+k)a_k=-\sum_{j=0}^{k-1}[(\alpha+j)p_{k-j}+q_{k-j}]a_j \quad (4.23)$$

To generate the first series solution, we take $\alpha=\alpha_1$: the solution of the indicial equation with the larger real part. Since F is a quadratic function with roots at α_1 and α_2, with $\mathrm{Re}[\alpha_2]\leq\mathrm{Re}[\alpha_1]$, it follows that $F(\alpha_1+k)\neq 0$ for any integer $k\geq 1$ We can therefore rearrange (4.23) to

$$a_k=-\frac{1}{F(\alpha_1+k)}\sum_{j=0}^{k-1}[(\alpha_1+j)p_{k-j}+q_{k-j}]a_j \quad (4.24)$$

and thus solve successively for all the coefficients a_1, a_2, ..., and we obtain one solution

$$y_1(x)=(x-x_0)^{\alpha_1}\sum_{k=0}^{\infty}a_k(x-x_0)^k. \quad (4.25)$$

Therefore at least one solution of (4.17) can always be expressed as a Frobenius series with indicial exponent $\alpha=\alpha_1$, and we call this the *first solution.*

4.2.3 The second solution Case I: $\alpha_1 - \alpha_2 \notin \mathbb{Z}$

For the *second solution*, we have to distinguish between several cases and sub-cases. The simplest case occurs when the indices α_1 and α_2 *do not differ by an integer* (so in particular they are not equal). In this case, $F(\alpha_2 + k) \neq 0$ for all $k \geq 1$, so we can solve (4.23) also with the second value of the exponent $\alpha = \alpha_2$. We call the coefficients the second solution b_n to distinguish from the previous coefficients a_k, and they satisfy the recurrence relations

$$b_k = -\frac{1}{F(\alpha_2 + k)} \sum_{j=0}^{k-1} [(\alpha_2 + j)p_{k-j} + q_{k-j}]b_j. \tag{4.26}$$

Thus, we obtain with no problems a second solution also as a Frobenius series, with indicial exponent α_2:

$$y_2(x) = (x - x_0)^{\alpha_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k. \tag{4.27}$$

4.2.4 Case II: $\alpha_1 = \alpha_2$

In the case of a double root we apparently only get one solution with the Frobenius method, and we have to multiply by logs to get a second solution (similar to the case of a double root in Cauchy–Euler). In particular, the second solution is of the form

$$y_2(x) = y_1(x) \log(x - x_0) + (x - x_0)^{\alpha_1} \sum_{k=0}^{\infty} b_k (x - x_0)^k, \tag{4.28}$$

where y_1 is the first solution (4.25).

The form of solution (4.28) can be derived using the so-called *derivative method*, which is outlined in §4.2.6. For the moment, we can at least verify that it works in principle by substituting (4.28) into (4.17). In doing so, note that, with $\mathfrak{L}$ defined by (4.17),

$$\mathfrak{L}\big[y_1(x) \log(x - x_0)\big] = \log(x - x_0)\mathfrak{L}y_1(x) + \frac{2}{(x - x_0)}\, y_1'(x) + \frac{p(x) - 1}{(x - x_0)^2}\, y_1(x) \tag{4.29}$$

and because $\mathfrak{L}y_1 = 0$, when (4.28) is substituted into (4.17), the logs vanish, and one can solve a sequence of recurrence relations for the coefficients b_k as above.

4.2.5 Case III: $\alpha_1 - \alpha_2$ a positive integer

If $\alpha_1 - \alpha_2$, where $N > 0$ is an integer, then we will potentially run into trouble in equation (4.26) when $k = N$. In this case, there are two sub-possibilities.

Case III(a): For $k = N$, the right-hand side of (4.26) is non-zero.

Then we have a contradiction, and the standard Frobenius solution method doesn't work. To get a second solution, we use the same form as in Case II, i.e.

$$y_2(x) = y_1(x) \log(x - x_0) + (x - x_0)^{\alpha_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k. \tag{4.30}$$

Again, when we substitute (4.30) into the ODE (4.17), the logs vanish and one obtains a set of recurrence relations that determine the coefficients b_k. Note that the indicial exponent for the second series in (4.30) is α_2, whereas y_1 is given by the Frobenius series using the exponent α_1.

Case III(b): When $k = N$, the right-hand side of RHS of (4.26) is zero.

In this case, there is no contradiction, but any choice for b_N will satisfy (4.26), i.e. b_N remains undetermined. The second solution therefore has Frobenius form

$$y_2(x) = (x - x_0)^{\alpha_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k, \tag{4.31}$$

where b_0 can be chosen to be $b_0 = 1$ (without loss of generality) and b_N is arbitrary. Since $\alpha_2 + N = \alpha_1$, changing b_N just corresponds to adding multiples of y_1 to (4.31).

Example 4.5. Find a series solution about the regular singular point $x = 0$ for the differential equation

$$4x^2 y''(x) + 4xy'(x) + (4x^2 - 1)y(x) = 0. \tag{4.32}$$

Step 1: *Assume a solution of form*

$$y(x) = x^{\alpha} \sum_{k=0}^{\infty} a_k x^k \tag{4.33}$$

with $a_0 \neq 0$. Compute the corresponding series for y', y'' by differentiating term by term.

Step 2: *Plug the series* (4.33) *into the ODE* (4.32) *and multiply everything out:*

$$\begin{aligned} 0 &= \underbrace{\sum_{k=0}^{\infty} 4(\alpha+k)(\alpha+k-1) a_k x^{\alpha+k}}_{4x^2y''} + \underbrace{\sum_{k=0}^{\infty} 4(\alpha+k) a_k x^{\alpha+k}}_{4xy'} - \underbrace{\sum_{k=0}^{\infty} a_k x^{\alpha+k}}_{y} + \underbrace{\sum_{k=0}^{\infty} 4 a_k x^{\alpha+k+2}}_{4x^2y} \\ &= \sum_{k=0}^{\infty} \left(4(\alpha+k)^2 - 1\right) a_k x^{\alpha+k} + \sum_{k=0}^{\infty} 4 a_k x^{\alpha+k+2}. \end{aligned} \tag{4.34}$$

Step 3: *The* indicial equation *comes from the balance at lowest order, in this case x^{α}:*

$$F(\alpha) = 4\alpha^2 - 1. \tag{4.35}$$

The indicial exponents are the roots of F, i.e.

$$\alpha_1 = \frac{1}{2}, \qquad \alpha_2 = -\frac{1}{2}. \tag{4.36}$$

Step 4: *Shift the terms in the series* (4.34) *so that* the exponents of x are the same *in each term. For this example, we need only shift the index in the last sum, so all the series have terms proportional to $x^{\alpha+k}$. Thus, by replacing k with $k-2$, we have*

$$\sum_{k=0}^{\infty} 4 a_k x^{\alpha+k+2} \equiv \sum_{k=2}^{\infty} 4 a_{k-2} x^{k+\alpha}, \tag{4.37}$$

and thus we obtain

$$0 = a_0 F(\alpha) x^{\alpha} + a_1 F(\alpha+1) x^{\alpha+1} + \sum_{k=2}^{\infty} \left[a_k F(\alpha+k) + 4a_{k-2}\right] x^{k+\alpha}. \tag{4.38}$$

We have chosen the α *so that the equation balances at* x^{α}*, and hence* a_0 *is free. Balancing at all other orders will determine the coefficients* a_k *for* $k \geq 1$*.*

Step 5: First series

Set $\alpha = \alpha_1 = 1/2$ *in* (4.38)*; note that*

$$F(\alpha_1 + k) = 4\left(\frac{1}{2} + k\right)^2 - 1 = 4k(k+1) \tag{4.39}$$

and thus we obtain

$$a_1 = 0, \qquad a_k = \frac{-1}{k(k-1)}\, a_{k-2} \quad k = 2,\, 3,\, \ldots. \tag{4.40}$$

Step 6: *Use the recursion relation* (4.40) *to determine a formula for* a_k *in terms of* a_0*. A good idea is to write out a few terms, and look for a pattern. Here, since* $a_1 = 0$*, we easily see that* $a_3 = a_5 = \cdots = 0$*, i.e. all the odd coeffients are zero, and we are left with*

$$\begin{aligned} a_2 &= \frac{-1}{2\cdot 3}\, a_0, \\ a_4 &= \frac{-1}{4\cdot 5}\, a_2 = \frac{1}{5\cdot 4\cdot 3\cdot 2}\, a_0, \\ &\ldots\ldots\ldots\ldots\ldots \\ a_{2k} &= \frac{(-1)^k a_0}{(2k+1)!}. \end{aligned} \tag{4.41}$$

Step 7: *Input the formula* (4.41) *for the coefficients to obtain the first solution:*

$$y_1(x) = a_0 x^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k}. \tag{4.42}$$

Step 8: Second series

Repeat the process for the second root $\alpha_2 = -1/2$*. In this case,* $\alpha_1 - \alpha_2 = 1 = N$ *is an integer, so we are in* Case III.

The coefficients b_k *in the second series satisfy*

$$0 = b_0 F(\alpha_2) x^{\alpha_2} + b_1 F(\alpha_2+1) x^{\alpha_2+1} + \sum_{k=2}^{\infty} \left[b_k F(\alpha_2+k) + 4b_{k-2}\right] x^{k+\alpha_2}. \tag{4.43}$$

The coefficient of x^{α_2}*, namely* $F(\alpha_2)$*, is zero by construction. At order* $x^{\alpha_2+N} = x^{\alpha_2+1}$*, we obtain* $F(1/2)b_1 = 0 \times b_1 = 0$*. There is no contradiction, and* b_1 *is arbitrary and can be set to zero: we are in* CaseIII(b).

Step 9: *Following the recursion forward with* $b_0 \neq 0$*, analogous computations to the above yield*

$$y_2(x) = b_0 x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k}. \tag{4.44}$$

Step 10: *The general solution is a linear combination of the two series solutions, i.e.*

$$y(x) = C_1 x^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} + C_2 x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}. \tag{4.45}$$

In this example, we can recognise the series for sine and cosine and thus express the solution in closed form. In fact, the general solution to (4.32) *(which is called* Bessel's equation *of order* 1/2*) is*

$$y(x) = C_1 \frac{\sin x}{\sqrt{x}} + C_2 \frac{\cos x}{\sqrt{x}}. \tag{4.46}$$

4.2.6 Derivative method

Here we discuss Case II, where α_1 is a double root of $F(\alpha)$, and give a brief justification for the form (4.28) of the series solution. Without loss of generality, let $a_0 = 1$. Suppose we solve (4.23) for the coefficients $a_1, a_2, \ldots$ with *arbitrary* α, i.e. with $F(\alpha)$ not generally equal to zero. Thus, each coefficient a_k is a function of α, and we can think of α as a parameter in the series

$$y(x;\alpha) = (x - x_0)^{\alpha} + \sum_{k=1}^{\infty} a_k(\alpha)(x - x_0)^{k+\alpha}. \tag{4.47}$$

The recurrence relation (4.23) ensures that the coefficient of $(x - x_0)^{\alpha+k-2}$ in $\mathfrak{L}y$ is zero for all $k \geq 1$, and we are just left with

$$\mathfrak{L}y(x;\alpha) = (x - x_0)^{\alpha-2} F(\alpha). \tag{4.48}$$

Since $F(\alpha_1) = 0$, it follows that $\mathfrak{L}y(x;\alpha_1) = 0$ and thus

$$y_1(x) = y(x;\alpha_1) = \sum_{0}^{\infty} a_k(\alpha_1)(x - x_0)^{\alpha_1+k} \tag{4.49}$$

is a solution (as we already know). Now the idea is to differentiate (4.48) with respect to α, then set $\alpha = \alpha_1$. Since $\mathfrak{L}$ has no dependence on α,

$$\begin{aligned} \mathfrak{L}\left[\frac{\partial}{\partial \alpha} y(x;\alpha)\right] &= \frac{\partial}{\partial \alpha}\left[\mathfrak{L}y(x;\alpha)\right] \\ &= \frac{\partial}{\partial \alpha}\left[(x - x_0)^{\alpha-2} F(\alpha)\right] \\ &= (x - x_0)^{\alpha-2} \log(\alpha) F(\alpha) + (x - x_0)^{\alpha-2} F'(\alpha). \end{aligned} \tag{4.50}$$

Since α_1 is a double root of F, the right-hand side of (4.50) is zero when $\alpha = \alpha_1$, and it follows that

$$y_2(x) = \left.\frac{\partial}{\partial \alpha} y(x;\alpha)\right|_{\alpha=\alpha_1} \tag{4.51}$$

satisfies $\mathfrak{L}y_2 = 0$. To get a more explicit form, calculate

$$\begin{aligned} \frac{\partial}{\partial \alpha} y(x;\alpha) &= \frac{\partial}{\partial \alpha}\left(\sum_{k=0}^{\infty} a_k(\alpha)(x - x_0)^{\alpha+k}\right) \\ &= \log(x - x_0) \sum_{k=0}^{\infty} a_k(\alpha)(x - x_0)^{\alpha+k} + \sum_{k=0}^{\infty} a_k'(\alpha)(x - x_0)^{\alpha+k} \end{aligned} \tag{4.52}$$

and set $\alpha = \alpha_1$ to get

$$y_2(x) = \log(x - x_0)y_1(x) + \sum_{k=0}^{\infty} b_k (x - x_0)^{\alpha_1 + k}, \tag{4.53}$$

in agreement with (4.28), where $b_k = a_k'(\alpha_1)$.

In principle, the derivative method allows us to determine the coefficients b_n in the second series solution, as outlined above. However, to do so we require a closed form for $a_k(\alpha)$ for general α. In practice, it is usually easier just to substitute in the appropriate form (4.28) of the series and compare coefficients.

4.2.7 More examples

Example 4.6. Find a series solution about $x = 0$ for the differential equation

$$x(x-1)y''(x) + 3xy'(x) + y(x) = 0. \tag{4.54}$$

First we divide through by $x(x-1)$ to obtain the standard form

$$y''(x) + \frac{3}{x-1}\,y'(x) + \frac{1}{x(x-1)}\,y(x) = 0. \tag{4.55}$$

Since $p(x) = 3x/(x-1)$ and $q(x) = x/(x-1)$ are analytic at $x = 0$, it is a regular singular point. Thus we expect to find at least one solution in the form of a Frobenius series.

If we try for a solution with the local behaviour of the form $y(x) \sim x^\alpha$ as $x \to 0$, then (4.54) implies that

$$-\alpha(\alpha-1)x^{\alpha-1} + \text{ higher order terms } = 0, \tag{4.56}$$

and we deduce that the indicial equation is

$$F(\alpha) = \alpha(\alpha - 1) = 0, \tag{4.57}$$

which has roots $\alpha_1 = 1$, $\alpha_2 = 0$.

More generally, by seeking the solution as a power series of the form

$$y(x) = x^\alpha \sum_{k=0}^{\infty} a_k x^k \tag{4.58}$$

we obtain

$$\underbrace{\sum_{k=0}^{\infty} -(k+\alpha)(k+\alpha-1)a_k x^{k+\alpha-1}}_{\text{series 1}} + \underbrace{\sum_{k=0}^{\infty} \left[(k+\alpha)(k+\alpha-1) + 3(k+\alpha) + 1\right] a_k x^{k+\alpha}}_{\text{series 2}} = 0. \tag{4.59}$$

Now, shift the index in series 2 so that the indices match series 1:

$$\text{series 2} = \sum_{k=1}^{\infty} \big[(k+\alpha-1)(k+\alpha-2) + 3(k+\alpha-1) + 1\big] a_{k-1} x^{k+\alpha-1}. \tag{4.60}$$

Now we can bring the two sums together and demand that the coefficients of $x^{k+\alpha-1}$ all vanish. The first term with $k = 0$ vaishes identically by the indicial equation (4.57). Simplifying the terms for $k > 0$, we obtain the recursion relation

$$(k+\alpha)(k+\alpha-1)a_k - (k+\alpha)^2 a_{k-1} = 0. \tag{4.61}$$

Note that the coefficient of a_k *is just* $F(k+\alpha)$, *as expected.*

On substituting $\alpha = \alpha_1 = 1$ *into* (4.61), *we obtain*

$$a_k = \frac{k+1}{k}\, a_{k-1}. \tag{4.62}$$

Without loss of generality setting $a_0 = 1$, *we obtain the simple formula* $a_k = k+1$, *and thus one solution to* (4.54) *is given by the series*

$$y_1(x) = \sum_{k=0}^{\infty}(k+1)x^{k+1} = \frac{x}{(1-x)^2}. \tag{4.63}$$

For a second solution, since $\alpha_1 - \alpha_2 = 1$ *is an integer, we are in* Case III, *and there may or may not be a Frobenius series solution. To find out, we seek a solution*

$$y_2 = x^{\alpha_2}\sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} b_k x^k. \tag{4.64}$$

Setting $\alpha = \alpha_2 = 0$ *in* (4.61), *we have*

$$(k-1)b_k = k b_{k-1}. \tag{4.65}$$

We immediately run into trouble, since we must take $b_0 \neq 0$, *and thus with* $k = 1$ *we get the contradiction* $0 \times b_1 = b_0 \neq 0$. *Hence the second Frobenious solution does not work: we are in* Case III(a), *and the form of the second solution is*

$$y_2(x) = y_1(x)\log(x) + \sum_{k=0}^{\infty} b_k x^k. \tag{4.66}$$

Example 4.6 illustrates that the indicial equation can be found just by considering the leading-order terms, without bothering to substitute in an entire series. In Example 4.6, once we have obtained one series solution $y_1(x) = x/(1-x)^2$, we can construct the other using reduction of order. Setting $y(x) = y_1(x)v(x)$ in (4.54), we find that v satisfies the ODE

$$v''(x) + \frac{(2-x)}{x(1-x)}\, v'(x) = 0, \tag{4.67}$$

which is easily integrated to give

$$v(x) = C_1 + C_2\left(\log(x) + \frac{1}{x}\right). \tag{4.68}$$

A second solution to (4.54) is thus given by

$$y_2(x) = y_1(x)\left(\log(x) + \frac{1}{x}\right) = y_1(x)\log(x) + \frac{1}{(1-x)^2}, \tag{4.69}$$

which indeed has the form (4.66) when expanded about $x = 0$.

Example 4.7. Find the form of series solutions about $x = 0$ for the differential equation

$$\sin^2(x)y'' - \sin(x)\cos(x)y' + y = 0. \tag{4.70}$$

We consider the functions

$$p(x) = -x\frac{\sin(x)\cos(x)}{\sin^2(x)}, \qquad q(x) = x^2\frac{1}{\sin^2(x)}. \tag{4.71}$$

As both p *and* q *are finite as* $x \to 0$ *(the singularities there are removable),* $x = 0$ *is a regular singular point. Note that*

$$\lim_{x\to 0} p(x) = -1, \qquad \lim_{x\to 0} q(x) = 1, \tag{4.72}$$

as can be obtained with L'Hôpital's rule. This implies that the leading terms in the power series expansions of p *and* q *are* $p_0 = -1$ *and* $q_0 = 1$*, and the indicial equation is*

$$F(\alpha) = \alpha(\alpha - 1) + p_0\alpha + q_0 = (\alpha - 1)^2 = 0. \tag{4.73}$$

Hence $\alpha = 1$ *is a repeated root.*

We conclude that one solution is of the form

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^{k+1} \tag{4.74a}$$

and a second solution is given by

$$y_2(x) = y_1(x)\log(x) + \sum_{k=0}^{\infty} b_k x^{k+1}. \tag{4.74b}$$

The coefficients $\{a_k,\, b_k\}$ *can in principle be computed by inserting the solution forms* (4.74) *into* (4.70) *and balancing coefficients, but we will not do so here.*

One can solve (4.70) exactly by spotting that $\sin x$ is a solution and then using reduction of order: this approach confirms that the local expansions (4.74) are indeed of the correct form.