6 Asymptotic analysis

These lecture notes are based on material written by Derek Moulton. *Please send any corrections or comments to* Peter Howell.

6.1 Introduction

A complex mathematical problem often cannot be solved exactly, but it may contain parameters that represent physical constants or quantities in the problem. If some of these parameters are very small or very large, it may be possible to derive approximate solutions to the problem. Doing this in a systematic manner is the subject of *asymptotic analysis*. In this section a basic framework is presented for the use of this approach. Asymptotic methods can be put on a rigorous footing, but we will content ourselves with an informal approach.

Example 6.1. Consider a pendulum, initially hanging vertically and set in motion with velocity V. *The angle $\theta(t)$ made by the pendulum with the vertical at time t satisfies the equation*

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0, \tag{6.1.1a}$$

where ℓ is the length of the pendulum and g is the acceleration due to gravity. The given initial state leads to the following initial conditions for θ:

$$\theta(0) = 0, \qquad \ell\dot{\theta}(0) = V. \tag{6.1.1b}$$

The problem (6.1.1) can *be solved exactly, but in a rather unpleasant form involving elliptic functions. Can we say anything about the how the solution depends on the sizes of the constants ℓ and V?*

The first step is to normalise the problem, i.e. to re-scale the variables to eliminate as many parameters as possible. The idea is that all of the variables and parameters in the normalised model should be dimensionless.

We can eliminate g/ℓ from (6.1.1a) *by defining a new time variable*

$$\tau = \left(\frac{g}{\ell}\right)^{1/2} t. \tag{6.1.2}$$

Note that g, ℓ and t have units of m^2/s, m *and* s, *respectively, so that τ is indeed* dimensionless. *The angle θ is already dimensionless, but nevertheless can be scaled to balance the left- and right-hand sides of* (6.1.1b), *i.e.*

$$\theta(t) = \alpha u(\tau), \tag{6.1.3}$$

where

$$\alpha = \frac{V}{\sqrt{\ell g}}. \tag{6.1.4}$$

Again, one can check that α is dimensionless.

The normalised version of the problem (6.1.1) *then reads*

$$\alpha\ddot{u}(\tau) + \sin\big(\alpha u(\tau)\big) = 0, \qquad u(0) = 0, \qquad \dot{u}(0) = 1. \tag{6.1.5}$$

Now we have collapsed all of the physical constants g, ℓ and V into the single dimensionless parameter α, and we can ask the question: how does the solution $u(\tau)$ of (6.1.5) *behave if α is very small or if α is very large?*

Example 6.1 illustrates how a process of *non-dimensionalisation* can produce a normalised mathematical problem containing a minimal number of dimensionless parameters that characterise the relative importance of the different physical effects in the problem. It then makes sense to ask what the approximate behaviour of solutions might be if a particular parameter is either very small or very large. More details on how to nondimensionalise a given physical problem can be found elsewhere and in Part B and C applied mathematical courses.

6.2 Asymptotic expansions

6.2.1 Order notation and twiddles

To start it is necessary to give a basic structure to describe approximations to a function when some parameter in the function becomes large or small. The following definitions allow the relative sizes of two different functions to be described. We consider two continuous real-valued functions $f(x)$ and $g(x)$, and compare their behaviours as x tends towards a particular value x_0 (often $x_0 = 0$ or ∞).

Definition 6.1. "Big O" notation
We write

$$f(x) = O\big(g(x)\big) \quad \text{as } x \to x_0 \qquad \text{if} \quad \exists A > 0 \text{ such that } |f(x)| < A|g(x)| \tag{6.2.1}$$

for all x sufficiently close to x_0.

We say that "f is of order g" to capture the idea that $f(x)$ and $g(x)$ are "roughly the same size" in the limit as $x \to x_0$.

Example 6.2.

(i) $\sin(2x) = O(x) \quad \text{as } x \to 0;$

(ii) $3x + x^3 = O(x) \quad \text{as } x \to 0;$

(iii) $\log x = O(x-1) \quad \text{as } x \to 1;$

(iv) $5x^2 + x^{-3} - \mathrm{e}^{-x} = O\left(x^2\right) \quad \text{as } x \to \infty.$

Definition 6.2. "Twiddles" notation
We write

$$f(x) \sim g(x) \quad \text{if} \quad \frac{f(x)}{g(x)} \to 1 \quad \text{as } x \to x_0. \tag{6.2.2}$$

This notation could be read as "f is asymptotic to g" or "f looks like g" as $x \to x_0$, and captures the idea of two functions being approximately equal in some limit.

Example 6.3.

(i) $\sin(2x) \sim 2x \quad \text{as } x \to 0;$

(ii) $x + \mathrm{e}^{-x} \sim x \quad \text{as } x \to \infty.$

Definition 6.3. "Little o" notation
We write

$$f(x) = o\big(g(x)\big) \quad \text{as } x \to x_0 \quad \text{if} \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0. \tag{6.2.3}$$

This notation captures the idea that f is "much smaller than" g in the limit as $x \to x_0$, and can also be written as $f(x) \ll g(x)$ or indeed $g(x) \gg f(x)$ as $x \to x_0$.

Example 6.4.

(i) $9x^2 - 4x^5 = o(x) \quad \text{as } x \to 0;$

(ii) $\dfrac{3}{x^2} - 3\mathrm{e}^{-x} = o(1/x) \quad \text{as } x \to \infty.$

Whenever using the order or twiddles notation, one should include in the statement what value x is tending to (though it is often implicit).

Example 6.5. Taylor's Theorem
A smooth function $f(x)$ may be expanded in a Taylor series and thus one may make statements such as:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + O\left((x - x_0)^2\right) \qquad \text{as } x \to x_0, \tag{6.2.4a}$$
$$f(x) = f(x_0) + (x - x_0)f'(x_0) + o\big((x - x_0)\big) \qquad \text{as } x \to x_0, \tag{6.2.4b}$$
$$f(x) = f(x_0) + (x - x_0)f'(x_0) + o\big((x - x_0)^{3/2}\big) \qquad \text{as } x \to x_0, \tag{6.2.4c}$$
$$f(x) \sim f(x_0) \qquad \text{as } x \to x_0, \tag{6.2.4d}$$
$$f(x) - f(x_0) \sim (x - x_0)f'(x_0) \qquad \text{as } x \to x_0. \tag{6.2.4e}$$

6.2.2 Asymptotic sequence and asymptotic expansion

In this course we are particularly interested in problems containing a small parameter, and we will therefore focus on the case $x_0 = 0$. We will follow convention by generally using the notation ϵ (rather than x) for the small parameter. Our aim then is to find the approximate behaviour of some function $f(\epsilon)$, say, in the limit as $\epsilon \to 0$.

Example 6.6.

(i) $\sin\big(\epsilon^{1/2}\big) \approx \epsilon^{1/2} - \dfrac{\epsilon^{3/2}}{6} + \cdots,$

(ii) $\tanh^{-1}(1 - \epsilon) \approx \dfrac{1}{2}\log\left(\dfrac{2}{\epsilon}\right) - \dfrac{\epsilon}{4} - \dfrac{\epsilon^2}{16} + \cdots,$

both in the limit as $\epsilon \to 0$.

If f is smooth, then one can express $f(\epsilon)$ as a Taylor expansion in powers of ϵ as $\epsilon \to 0$, as in Example 6.5. However, for a unbounded or non-smooth functions, integer powers of ϵ might not be appropriate to capture the local behaviour, as illustrated by Example 6.6. In general, we might want to write

$$f(\epsilon) \approx \sum_k a_k \phi_k(\epsilon), \tag{6.2.5}$$

where $\phi_k(\epsilon)$ are suitable *gauge functions*. For such a series to provide a useful approximation to the function f, we would expect the terms in the expansion to get successively smaller with increasing k, and this motivates the following definition.

Definition 6.4. *A set of functions $\{\phi_k(\epsilon)\}_{k=0,1,2,\ldots}$ is an* asymptotic sequence *as $\epsilon \to 0$ if $\phi_{k+1}(\epsilon) = o\big(\phi_k(\epsilon)\big)$ as $\epsilon \to 0$, i.e. each term in the sequence is of smaller magnitude than the previous term.*

Example 6.7. *Here are some examples of asymptotic sequences:*

(i) $\{1,\, \epsilon,\, \epsilon^2,\, \epsilon^3,\, \cdots\}$,

(ii) $\left\{1,\, \epsilon^{1/2},\, \epsilon,\, \epsilon^{3/2},\, \cdots\right\}$,

(iii) $\{1,\, \epsilon,\, \epsilon\log\epsilon,\, \epsilon,\, \epsilon^2\log\epsilon,\, \cdots\}$.

Definition 6.5. *A function $f(\epsilon)$ has an* asymptotic expansion *of the form*

$$f(\epsilon) \sim \sum_k a_k\phi_k(\epsilon) \quad \text{as } \epsilon \to 0 \tag{6.2.6}$$

if

(i) *the gauge functions ϕ_k for an asymptotic sequence, i.e. $\phi_{k+1}(\epsilon) \ll \phi_k(\epsilon)$ for all k;*

(ii) $f(\epsilon) - \sum_{k=0}^{N} a_k\phi_k(\epsilon) \ll \phi_N(\epsilon)$ *for all $N = 0,\, 1,\, \ldots$,*

as $\epsilon \to 0$.

Property (i) ensures that *the terms in the expansion get successively smaller*, and property (ii) ensures that *the approximation gets more accurate the more terms are included in the expansion.*

The definition of an asymptotic expansion differs crucially from that for a *convergent* series. For a convergent series of the form

$$f(\epsilon) = \sum_{k=0}^{\infty} a_k\phi_k(\epsilon), \tag{6.2.7}$$

we require that the partial sum

$$f_N(\epsilon) = \sum_{k=0}^{N} a_k\phi_k(\epsilon), \tag{6.2.8}$$

converges to $f(\epsilon)$ as $N \to \infty$, with ϵ held fixed. For an asymptotic expansion

$$f(\epsilon) \sim \sum_k a_k\phi_k(\epsilon), \tag{6.2.9}$$

we instead require that the partial sum (6.2.8) converges asymptotically to $f(\epsilon)$ as $\epsilon \to 0$, with N held fixed. In fact, an asymptotic expansion may well *diverge* as $N \to \infty$ (i.e. have radius of convergence equal to zero) but still be useful and perfectly well defined by Definition 6.5.

Elementary properties of asymptotic expansions include the following.

1. *Given* a particular choice of gauge functions $\{\phi_k\}$, the coefficients a_k are unique.

This can easily be proved by induction on N. Note that the gauge functions themselves are not unique, for example,

$$\begin{aligned} \tan\epsilon &\sim \epsilon + \frac{1}{3}\epsilon^3 + \frac{2}{15}\epsilon^5 + \cdots \\ &\sim \sin\epsilon + \frac{1}{2}\sin^3\epsilon + \frac{3}{8}\sin^5\epsilon + \cdots. \end{aligned} \tag{6.2.10}$$

Usually we use the simplest choice, namely powers of ϵ, or possibly exponentials or logs.

2. The function defines the expansion but not vice versa.

For example, if $\phi_k(\epsilon) = \epsilon^k$ for $k = 0, 1, 2, \ldots$, then

$$\frac{1}{1-\epsilon} \sim 1 + \epsilon + \epsilon^2 + \cdots \quad \text{as } \epsilon \to 0 \tag{6.2.11a}$$

but also

$$\frac{1}{1-\epsilon} + \mathrm{e}^{-1/\epsilon} \sim 1 + \epsilon + \epsilon^2 + \cdots \quad \text{as } \epsilon \to 0. \tag{6.2.11b}$$

In other words, we have two different functions with the same asymptotic expansion. This occurs because (for $0 < \epsilon \ll 1$)

$$\mathrm{e}^{-1/\epsilon} = o\left(\epsilon^k\right) \quad \text{for all } k, \tag{6.2.12}$$

and $\mathrm{e}^{-1/\epsilon}$ is said to be *exponentially small* or *transcendentally small.*

6.3 Approximate roots of algebraic equations

To start using asymptotic methods consider the problem of finding the roots of an algebraic equation containing a small parameter. To focus ideas, first we consider some simple cases where the exact roots can be easily found.

Example 6.8. Solve approximately the quadratic equation

$$x^2 + \epsilon x - 1 = 0 \tag{6.3.1}$$

in the limit as $\epsilon \to 0$.
Exact solution: *Here we can use the quadratic formula to get the exact solutions*

$$x = \frac{1}{2}\left(-\epsilon \pm \sqrt{4+\epsilon^2}\right). \tag{6.3.2}$$

A binomial expansion of the square root yields the following approximations for the two roots:

$$x^+ \sim 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + O\left(\epsilon^4\right), \tag{6.3.3a}$$

$$x^- \sim -1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + O\left(\epsilon^4\right), \tag{6.3.3b}$$

both as $\epsilon \to 0$. Now the question is, could we have derived the approximate solutions (6.3.3) *directly from the equation* (6.3.1), *without finding the exact solutions first?*
Asymptotic approach: *Since equation* (6.3.1) *contains only ϵ, and no other (e.g. fractional) powers of ϵ, we assume that the solution for x may be expressed as an asymptotic expansion of the form*

$$x \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots \quad \textit{as } \epsilon \to 0. \tag{6.3.4}$$

We substitute (6.3.4) *into* (6.3.1) *to obtain*

$$\begin{aligned} 0 &\sim \left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right)^2 + \epsilon\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right) - 1 \\ &\sim x_0^2 + 2x_0 x_1 \epsilon + \left(x_1^2 + 2x_0 x_2\right)\epsilon^2 + \left(2x_1 x_2 + 2x_0 x_3\right)\epsilon^3 + \cdots \epsilon\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right) - 1. \end{aligned} \tag{6.3.5}$$

Since this must hold for all *ϵ, and we have assumed that $x_0, x_1, \ldots$* are all independent of *ϵ, we conclude that equality must hold independently for each power of ϵ. Hence, we equate the coefficients of each power of ϵ to solve successively for $x_0, x_1, \ldots$.*

Considering the first few powers, we get:

$$O(1): \quad x_0^2 - 1 = 0, \quad \Rightarrow \quad x_0 = \pm 1, \tag{6.3.6a}$$
$$O(\epsilon): \quad 2x_0x_1 + x_0 = 0, \quad \Rightarrow \quad x_1 = -\frac{1}{2}, \tag{6.3.6b}$$
$$O\left(\epsilon^2\right): \quad 2x_0x_2 + x_1^2 + x_1 = 0, \quad \Rightarrow \quad x_2 = \frac{1}{8x_0} = \pm\frac{1}{8}, \tag{6.3.6c}$$
$$O\left(\epsilon^3\right): \quad 2x_0x_3 + 2x_1x_2 + x_2 = 0, \quad \Rightarrow \quad x_3 = 0, \tag{6.3.6d}$$

and so on. Thus we have obtained the first few terms in asymptotic expansions for each of the two roots of (6.3.1)*, namely*

$$x \sim \pm 1 - \frac{1}{2}\,\epsilon \pm \frac{1}{8}\,\epsilon^2 + O\left(\epsilon^4\right), \tag{6.3.7}$$

which clearly agrees with the exact solution (6.3.3)*.*

Example 6.9. Solve approximately the quadratic equation

$$\epsilon x^2 + x - 1 = 0 \tag{6.3.8}$$

in the limit as $\epsilon \to 0$.
Exact solution: *Again we can use the quadratic formula to get the exact solutions*

$$x = \frac{1}{2\epsilon}\left(-1 \pm \sqrt{1+4\epsilon}\right), \tag{6.3.9}$$

and expansion of the square root yields the following approximations for the two roots:

$$x^+ \sim 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + O\left(\epsilon^4\right), \tag{6.3.10a}$$
$$x^- \sim -\frac{1}{\epsilon} - 1 + \epsilon - 2\epsilon^2 + O\left(\epsilon^3\right). \tag{6.3.10b}$$

Now we try to get the roots directly from equation (6.3.8)*.*
Asymptotic approach. First attempt: *It is reasonable to expect that the leading-order solution as $\epsilon \to 0$ could be found by just setting $\epsilon = 0$ in* (6.3.8)*. This approach gives $x \sim 1$ as a first approximation, which indeed agrees with the first root* (6.3.10a) *at lowest order in ϵ. We can then obtain an improved approximation by hypothesising that x can be expressed as an asymptotic expansion in powers of ϵ, i.e.*

$$x \sim 1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots \quad \text{as } \epsilon \to 0. \tag{6.3.11}$$

We substitute (6.3.11) *into the original equation* (6.3.8) *to get*

$$0 \sim \epsilon\left(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right)^2 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots. \tag{6.3.12}$$

As in Example 6.8*, we equate the coefficients of each power of ϵ to solve successively for x_1, x_2, ...*
Considering the first few powers, we get:

$$O(\epsilon): \quad 1 + x_1 = 0, \quad \Rightarrow \quad x_1 = -1, \tag{6.3.13a}$$
$$O\left(\epsilon^2\right): \quad 2x_1 + x_2 = 0, \quad \Rightarrow \quad x_2 = 2, \tag{6.3.13b}$$
$$O\left(\epsilon^3\right): \quad x_1^2 + 2x_2 + x_3 = 0, \quad \Rightarrow \quad x_3 = -5, \tag{6.3.13c}$$

and so on. Hence we can systematically improve the approximation of the root near $x = 1$, and evidently we have managed to reproduce the expansion (6.3.10a)*.*
Second attempt: *The previous approach successfully produced an asymptotic expansion for the positive root x^+. But since* (6.3.8) *is a quadratic equation, we know that it has another root, which our method seems to have missed.*

Note that the root (6.3.11) *near* $x = 1$ *has been found by considering a* dominant balance *between two of the three terms in* (6.3.8)*, namely* x *and* 1*, while treating the third term* ϵx^2 *as a small correction, i.e.*

$$\underbrace{\epsilon x^2}_{\text{small}} + \underbrace{x - 1}_{\text{balance}} = 0. \tag{6.3.14}$$

To approximate the other root, we need to consider other possible balances between different terms in equation (6.3.8)*.*

Suppose we try to balance the terms $\epsilon^2 x$ *and* 1 *in* (6.3.8)*, which suggests that* $x = O\left(\epsilon^{-1/2}\right)$*. This choice would give the following sizes for the terms:*

$$\underbrace{\epsilon x^2}_{O(1)} + \underbrace{x}_{O\left(\epsilon^{-1/2}\right)} - \underbrace{1}_{O(1)} = 0. \tag{6.3.15}$$

Now we have a problem: the first and third terms balance, but the second term is much bigger than either of them. To get a dominant balance*, we need to ensure that the balanced terms are the dominant terms in the equation, and* (6.3.15) *fails this requirement.*

Third attempt: *The remaining possibility is to balance the terms* ϵx^2 *and* x *in* (6.3.8)*, i.e. to suppose that* $x = O\left(\epsilon^{-1}\right)$*. Then comparing the sizes of the terms in* (6.3.8)*, we get*

$$\underbrace{\epsilon x^2}_{O(\epsilon^{-1})} + \underbrace{x}_{O(\epsilon^{-1})} - \underbrace{1}_{O(1)} = 0. \tag{6.3.16}$$

This choice does give a dominant balance: when the first two terms are the same order, they are indeed much bigger than the third term.

Now we know this balance works, we use the scaling $x = \epsilon^{-1} y$*, with* $y = O(1)$*, to reflect the anticipated size of* x*; then* (6.3.8) *is transformed to*

$$\frac{y^2}{\epsilon} + \frac{y}{\epsilon} - 1 = 0 \quad \Leftrightarrow \quad y^2 + y - \epsilon = 0. \tag{6.3.17}$$

Now letting $\epsilon \to 0$ *in* (6.3.17)*, we get a sensible balance between the first two terms, but there seem to be two possible choices for* y*, namely* $y \sim -1$ *or* $y \sim 0$*. However, assuming that we have scaled the equation correctly, the desired root should have* $y = O(1)$*, so we ignore the second option (which in fact just reproduces the root* x_+ *that we have already found).*

We therefore seek the solution to (6.3.17) *as an asymptotic expansion of the form*

$$y \sim -1 + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \cdots \quad \text{as } \epsilon \to 0. \tag{6.3.18}$$

Substitution of (6.3.18) *into* (6.3.17) *leads to*

$$0 \sim \left(-1 + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \cdots\right)\left(\epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \cdots\right) - \epsilon, \tag{6.3.19}$$

after some simplification by writing $y^2 + y = y(y+1)$*. As above, this equation must be satisfied at every order in* ϵ*, and we can solve successively for the coefficients as follows:*

$$O(\epsilon): \quad -y_1 - 1 = 0, \quad \Rightarrow \quad y_1 = -1, \tag{6.3.20a}$$
$$O\left(\epsilon^2\right): \quad y_1^2 - y_2 = 0, \quad \Rightarrow \quad y_2 = 1, \tag{6.3.20b}$$
$$O\left(\epsilon^3\right): \quad 2y_1 y_2 - y_3 = 0, \quad \Rightarrow \quad y_3 = -2, \tag{6.3.20c}$$

and so on. We have thus constructed the approximate solution for y*, namely*

$$y \sim -1 - \epsilon + \epsilon^2 - 2\epsilon^3 + \cdots \quad \text{as } \epsilon \to 0, \tag{6.3.21}$$

and by rescaling $x = y/\epsilon$*, we see that we have successfully obtained the second root* x_- *given by* (6.3.10b)*.*

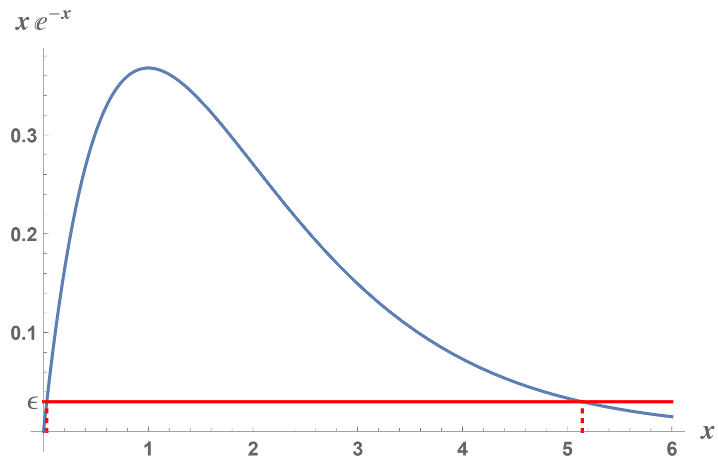

Figure 6.1: The function $x\mathrm{e}^{-x}$ plotted versus x, indicating two roots to equation (6.3.22) with $0 < \epsilon \ll 1$.

In Example 6.8, we can find both roots of equation (6.3.1) as regular asymptotic expansions in integer powers of ϵ, without any rescaling of x. In contrast, in Example 6.9, by seeking a regular expansion, we only manage to obtain one root; to find the other we have to rescale x appropriately. Consequently, one of the roots of equation (6.3.8) *diverges* like $1/\epsilon$ as $\epsilon \to 0$. This occurs because setting $\epsilon = 0$ reduces the degree of (6.3.8) from a quadratic to a linear equation, and thus reduces the number of roots from two to one. It is necessary to rescale x to reintroduce the quadratic term ϵx^2 at leading order to recover the second root. A so-called *singular perturbation* is said to occur when setting $\epsilon = 0$ reduces the degree of the problem, and thus the number of solutions that the problem possesses.

Example 6.9 illustrates the following general procedure to find an approximate solution x of an algebraic equation of the form $F(x; \epsilon) = 0$ containing a small parameter ϵ.

1. Scale the variable(s) to get a dominant balance, i.e. so that at least two of the terms (i) balance and (ii) are much bigger than the remaining terms in the equation.
2. Plug in an asymptotic expansion for x. Usually the form of the expansion is clear from the form of the equation (though see below an example where it isn't so clear).
3. By equating the terms multiplying each power of ϵ in the equation, obtain the coefficients in the expansion.
4. Repeat for any other possible dominant balances in the equation to obtain approximations for other roots.

We next try to use the same ideas to solve an equation where there is no exact solution to guide us.

Example 6.10. Find an asymptotic expansion for all the roots of

$$x\mathrm{e}^{-x} = \epsilon \quad \text{as } \epsilon \to 0. \tag{6.3.22}$$

Figure 6.1 *shows a plot of* $x\mathrm{e}^{-x}$ *versus* x*. For small, positive values of* ϵ*, we expect there to be two roots* x *of* (6.3.22)*: one close to* $x = 0$ *and one with* x *large.* [**Exercise:** *show that there exist two roots if* $\epsilon < \mathrm{e}^{-1}$.]

We consider the smaller root first. When x is small, we have $\mathrm{e}^{-x} = O(1)$ and, to balance the left- and right-hand sides of (6.3.22)*, we should therefore scale x with ϵ. We set $x = \epsilon y$, with y assumed to be $O(1)$, and equation* (6.3.22) *can then be written as*

$$y = \mathrm{e}^{\epsilon y} \sim 1 + \epsilon y + \frac{\epsilon^2 y^2}{2} + \frac{\epsilon^3 y^3}{6} + \cdots \quad \text{as } \epsilon \to 0. \tag{6.3.23}$$

The Maclaurin expansion of the right-hand side is valid given our hypothesis that $y = O(1)$.

Now we pose an asymptotic expansion for y: given that only integer powers of ϵ appear in equation (6.3.23)*, it is reasonable to assume that y may be expanded in the form*

$$y \sim y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots \sim 1 + \epsilon\left(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right) + \frac{1}{2}\epsilon^2\left(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right)^2 + \cdots. \tag{6.3.24}$$

We can then easily determine the coefficients:

$$y_0 = 1, \tag{6.3.25a}$$
$$y_1 = y_0 = 1, \tag{6.3.25b}$$
$$y_2 = y_1 + \frac{1}{2}y_0^2 = \frac{3}{2}, \tag{6.3.25c}$$

and so on, and therefore the smaller root of (6.3.22) *is given by the asymptotic expansion*

$$x \sim \epsilon + \epsilon^2 + \frac{3}{2}\epsilon^3 + \ldots \quad \text{as } \epsilon \to 0. \tag{6.3.26}$$

An asymptotic expansion for the larger root of (6.3.22) *is a lot harder to find. As a first step, we take logs of both sides of* (6.3.22) *to get*

$$x - \log x = -\log\epsilon = |\log\epsilon|. \tag{6.3.27}$$

> **Health warning:** *examples like this with logs are notoriously awkward: the solution of the apparantly innocuous algebraic equation* (6.3.27) *is just about as bad as one will ever encounter!*

Since ϵ is assumed to be very small (and positive), $\log\epsilon$ is large and negative, with $|\log\epsilon| \to \infty$ as $\epsilon \to 0$. To satisfy (6.3.27)*, x will need to be large, in which case $x \gg \log x$. To get a balance in* (6.3.27)*, we therefore scale $x = |\log\epsilon| y$ to get*

$$|\log\epsilon| y - \log(|\log\epsilon| y) = |\log\epsilon| \quad \Leftrightarrow \quad y - \frac{\log(|\log\epsilon|)}{|\log\epsilon|} - \frac{\log y}{|\log\epsilon|} = 1. \tag{6.3.28}$$

The difficulty here is that we can't assume a known form *of the asymptotic expansion for y and then just solve for the coefficients: it is not obvious in advance what gauge functions we should use. So let us just pose a general expansion of the form*

$$y \sim 1 + \phi_1(\epsilon) + \phi_2(\epsilon) + \cdots, \tag{6.3.29}$$

assuming only that $\cdots \ll \phi_2 \ll \phi_1 \ll 1$, and try to calculate what $\phi_1, \phi_2, \ldots$ should be. Note that (6.3.29) *gives*

$$\log y \sim (\phi_1 + \phi_2 + \cdots) - \frac{1}{2}(\phi_1 + \phi_2 + \cdots)^2 + \cdots \sim \phi_1 \tag{6.3.30}$$

to lowest order. Rearranging (6.3.28)*, we therefore obtain*

$$\underbrace{y - 1}_{\sim\phi_1} - \underbrace{\frac{\log y}{|\log\epsilon|}}_{\sim\phi_1/|\log\epsilon|} = \frac{\log(|\log\epsilon|)}{|\log\epsilon|}. \tag{6.3.31}$$

We observe that the first term dominates the second, and obtain a balance in (6.3.31) *by choosing*

$$\phi_1(\epsilon) = \frac{\log(|\log\epsilon|)}{|\log\epsilon|}. \tag{6.3.32}$$

Indeed this does give $\phi_1 \ll 1$*, in the sense that* $\phi_1(\epsilon) \to 0$ *as* $\epsilon \to 0$*, so our assumed form of the expansion* (6.3.29) *is self-consistent (so far at least).*

Again we rearrange (6.3.31) *to*

$$\underbrace{y - 1 - \phi_1}_{\sim\phi_2} = \underbrace{\frac{\log y}{|\log\epsilon|}}_{\sim\phi_1/|\log\epsilon|}, \tag{6.3.33}$$

and a leading-order balance is now obtained by choosing

$$\phi_2(\epsilon) = \frac{\phi_1(\epsilon)}{|\log\epsilon|} = \frac{\log(|\log\epsilon|)}{|\log\epsilon|^2}. \tag{6.3.34}$$

Again we can verify that $\phi_2 \ll \phi_1$*, i.e. that* $\phi_2(\epsilon)/\phi_1(\epsilon) \to 0$ *as* $\epsilon \to 0$*, so that our expansion is self-consistent. We thus get the early terms in an expansion for the larger root of* (6.3.22)*, namely*

$$x \sim |\log\epsilon| + \log(|\log\epsilon|) + \frac{\log(|\log\epsilon|)}{|\log\epsilon|} + \cdots \quad \text{as } \epsilon \to 0. \tag{6.3.35}$$

Exercise: *Show that the next term in the expansion is of order* $\big(\log(|\log\epsilon|)/|\log\epsilon|\big)^2$.

6.4 Regular perturbations in ODEs

We have shown how to use asymptotic methods to systematically approximate the roots of algebraic and transcendental equations. Now we explore how the same ideas may be used to find approximate solutions to ODEs.

Example 6.11. Find the approximate solution $y(x)$ of the following problem when $0 < \epsilon \ll 1$:

$$y''(x) = -\frac{1}{1+\epsilon y(x)^2}, \quad 0 < x < 1, \qquad y(0) = y(1) = 0. \tag{6.4.1}$$

The solution $y(x;\epsilon)$ *depends on both* x *and* ϵ*. Since the problem* (6.4.1) *contains only* ϵ*, and no other powers or functions of* ϵ*, it is reasonable to assume that the solution may be expressed as an asymptotic expansion in integer powers of* ϵ*, i.e.*

$$y(x;\epsilon) \sim y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots. \tag{6.4.2}$$

Putting this into the ODE (6.4.1)*, we get*

$$\begin{aligned} y_0'' + \epsilon y_1'' + \cdots &= -\frac{1}{1+\epsilon(y_0+\epsilon y_1+\cdots)^2} \\ &\sim -1 + \epsilon y_0^2 + \cdots, \end{aligned} \tag{6.4.3}$$

with boundary conditions

$$0 = y(0,\epsilon) \sim y_0(0) + \epsilon y_1(0) + \cdots, \qquad 0 = y(1,\epsilon) \sim y_0(1) + \epsilon y_1(1) + \cdots. \tag{6.4.4}$$

By setting in turn the coefficient of each power of ϵ to zero, we get

$$O(1): \qquad y_0'' = -1, \quad y_0(0) = y_0(1) = 0$$

$$\Rightarrow \quad y_0(x) = \frac{1}{2}\,x(1-x), \tag{6.4.5a}$$

$$O(\epsilon): \qquad y_1''(x) = y_0(x)^2 = \frac{1}{4}\,x^2(1-x)^2, \quad y_1(0) = y_1(1) = 0$$

$$\Rightarrow \quad y_1(x) = -\frac{1}{240}\,x(1-x)\left(2x^4 - 4x^3 + x^2 + x + 1\right), \tag{6.4.5b}$$

and so on.

Example 6.12. Small oscillations of a pendulum

Let us return to the problem (6.1.5) *from* Example 6.1*, in the limit where the dimensionless parameter α, which measures the strength of the initial impulse, is small. To cast the problem in a more familiar form, set $\alpha = \epsilon \ll 1$ and $u(\tau) = y(x)$ so the problem reads*

$$y''(x) + \frac{\sin\big(\epsilon y(x)\big)}{\epsilon} = 0, \qquad y(0) = 0, \qquad y'(0) = 1. \tag{6.4.6}$$

Note that

$$\frac{\sin(\epsilon y)}{\epsilon} \sim y - \frac{1}{6}\,\epsilon^2 y^3 + \frac{1}{120}\,\epsilon^4 y^5 + \cdots \quad \text{as } \epsilon \to 0, \tag{6.4.7}$$

and the problem (6.4.6) *therefore contains only* even *powers of ϵ. It follows that we can seek the solution for y as an asymptotic expansion of the form*

$$y(x;\epsilon) \sim y_0(x) + \epsilon^2 y_2(x) + \epsilon^4 y_4(x) + \cdots \quad \text{as } \epsilon \to 0. \tag{6.4.8}$$

(If we included intermediate terms like $\epsilon y_1(x)$ in the expansion (6.4.8)*, then on substitution into* (6.4.6) *we would find that they are identically zero.)*

Now we substitute (6.4.8) *into* (6.4.6) *and equate the coefficients of each power of ϵ as usual. At leading order we have the problem*

$$y_0'' + y_0 = 0, \qquad y_0(0) = 0, \qquad y_0'(0) = 1, \tag{6.4.9}$$

whose solution is given by

$$y_0(x) = \sin x. \tag{6.4.10}$$

At order ϵ^2, we get

$$y_2'' + y_2 = \frac{y_0^3}{6}, \qquad y_2(0) = 0, \qquad y_2'(0) = 0. \tag{6.4.11}$$

The right-hand side of (6.4.11) *can be written in the form*

$$\frac{1}{6}\,\sin^3(x) = \frac{1}{8}\,\sin(x) - \frac{1}{24}\,\sin(3x), \tag{6.4.12}$$

and we thus find the general solution for y_2 to be

$$y_2(x) = \frac{1}{192}\,\sin(3x) - \frac{1}{16}\,x\cos(x) + c_1\sin(x) + c_2\cos(x). \tag{6.4.13}$$

The integration constants are determined by applying the initial conditions, and thus we obtain

$$y_2(x) = \frac{3}{64}\,\sin(x) + \frac{1}{192}\,\sin(3x) - \frac{1}{16}\,x\cos(x). \tag{6.4.14}$$

The asymptotic expansion of the solution of the problem (6.4.6) *is thus given by*

$$y(x;\epsilon) \sim \sin(x) + \epsilon^2\left[\frac{3}{64}\,\sin(x) + \frac{1}{192}\,\sin(3x) - \frac{1}{16}\,x\cos(x)\right] + \cdots \tag{6.4.15}$$

as $\epsilon \to 0$.

Example 6.12 illustrates a potential difficulty that may be encountered when we try to write a function of *two* variables $y(x;\epsilon)$ as an asymptotic expansion in the limit $\epsilon \to 0$. The approximate solution (6.4.15) is a valid asymptotic expansion provided each term in the series is much smaller than the previous terms. This is certainly true if $x = O(1)$ and $\epsilon \ll 1$, but what happens when x gets very large? Eventually, when $x = O\left(1/\epsilon^2\right)$, the term proportional to $\epsilon^2 x$ becomes the same order as the leading-order term, and the expansion (6.4.15) ceases to be asymptotic. When x becomes sufficiently large, the expansion (6.4.15) is said to become *nonuniform*. In this example, the nonuniformity arises from the *secular term* proportional to $x\cos(x)$ in the solution for $y_2(x)$, which itself was a consequence of the forcing term proportional to $\sin(x)$ on the right-hand side of (6.4.11). In general, in problems like (6.4.11), we expect to find a secular term in the solution whenever the right-hand side contains a term that is in the complentary function (i.e. in the kernel of the differental operator on the left-hand side).

One can modify the solution (6.4.15) to a form that is valid for larger values of x by using the *method of multiple scales* — see §6.7.3 for a simple implementation of the method or C5.5 Perturbation Methods for the more general version. For the moment we consider another example where taking an infinite interval for the independent variable leads to trouble.

Example 6.13. Find the approximate solution of the IVP

$$y'(x) = y(x) - \epsilon y(x)^2, \quad x > 0, \qquad y(0) = 1, \tag{6.4.16}$$

as a regular asymptotic expansion in the limit $\epsilon \to 0$.

Writing the solution as an asymptotic expansion

$$y(x;\epsilon) \sim y_0(x) + \epsilon y_1(x) + \cdots, \tag{6.4.17}$$

and equating powers of ϵ in the usual way gives us

$$y_0(x) = \mathrm{e}^x, \tag{6.4.18}$$

and then

$$y_1'(x) = y_1(x) - \mathrm{e}^{2x}, \quad y_1(0) = 0 \qquad \Rightarrow \quad y_1(x) = \mathrm{e}^x - \mathrm{e}^{2x}. \tag{6.4.19}$$

We thus obtain the following asymptotic expansion for the solution:

$$y(x;\epsilon) \sim \mathrm{e}^x + \epsilon\left(\mathrm{e}^x - \mathrm{e}^{2x}\right) + \cdots \quad \text{as } \epsilon \to 0. \tag{6.4.20}$$

Now we see that the expansion becomes nonuniform when $\epsilon \mathrm{e}^{2x} \sim \mathrm{e}^x$, i.e. when $x = O\left(|\log\epsilon|\right)$.

In this case, we can solve the simple ODE (6.4.16) *exactly to get*

$$y(x;\epsilon) = \frac{\mathrm{e}^x}{1 + \epsilon\left(\mathrm{e}^x - 1\right)}. \tag{6.4.21}$$

Expansion of the solution (6.4.21) *in powers of ϵ indeed reproduces the approximation* (6.4.20)*, assuming that $x = O(1)$. However, the exact solution* (6.4.21) *satisfies $y(x) \to 1/\epsilon$ as $x \to \infty$, while the approximate solution* (6.4.20) *suggests that $y(x)$ grows without bound. Evidently the asymptotic approximation is valid only if x is not too large (specifically if $x \ll |\log\epsilon|$), and a different approach would be needed to approximate the solution for larger value of x. [Try substituting $x = \log(1/\epsilon) + X$ into* (6.4.21) *before expanding in powers of ϵ.]*

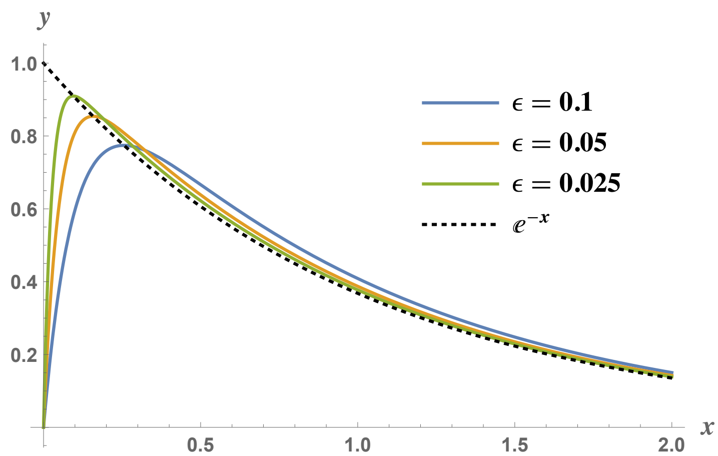

Figure 6.2: The function $y(x;\epsilon)$ given by (6.5.3) plotted versus x with three different values of ϵ. The leading-order outer solution e^{-x} is plotted as a black dotted curve.

6.5 Boundary layers

6.5.1 A first example

The solution of an ODE like (6.4.16), containing a parameter ϵ, is a function of *two* variables, namely ϵ and the independent variable x of the ODE. To obtain an approximate solution when ϵ is small, our starting point is generally to seek the solution as a regular asymptotic expansion of the form $y(x;\epsilon) \sim y_0(x) + \epsilon y_1(x) + \cdots$. However, the previous examples demonstrate that such an expansion may only be valid for a limited range of values of x. This may reduce the usefulness of the approximation. Even worse, it is not even clear how to determine the solution uniquely if a boundary condition is imposed in a region where the asymptotic expansion is not valid, as illustrated by the following simple example.

Example 6.14. Find the approximate solution of the IVP

$$\epsilon y'(x) + y(x) = \mathrm{e}^{-x}, \quad x > 0, \qquad y(0) = 0. \tag{6.5.1}$$

If we seek the solution as a regular asymptotic expansion of the form $y \sim y_0 + \epsilon y_1 + \cdots$*, then we find*

$$\begin{aligned} y_0(x) &= \mathrm{e}^{-x}, \\ y_1(x) &= -y_0'(x) = \mathrm{e}^{-x}, \end{aligned} \tag{6.5.2}$$

and so on. The problem is that we can never satisfy the boundary condition $y(0) = 0$*!*

The difficulty that in Example 6.14 occurs because the small parameter ϵ multiplies the highest derivative in the problem. In the limit $\epsilon \to 0$, the ODE (6.5.1) reduces to an *algebraic* equation, namely $y(x) \sim \mathrm{e}^{-x}$, and it becomes impossible to impose any initial condition.

The *exact solution* of (6.5.1) is given by

$$y(x;\epsilon) = \frac{\mathrm{e}^{-x}}{1-\epsilon} - \frac{\mathrm{e}^{-x/\epsilon}}{1-\epsilon}, \tag{6.5.3}$$

which is plotted versus x for small but nonzero values of ϵ in Figure 6.2. We see that $y(x) \sim \mathrm{e}^{-x}$ *does* provide a good approximation to the exact solution for nearly all values of x.

However, e^{-x} stops being a good approximation to $y(x)$ in a narrow region, called a *boundary layer*, close to $x = 0$, where the solution rapidly adjusts to satisfy the boundary condition $y(0) = 0$. Examining the exact solution (6.5.3), we can see that the rapid variation near $x = 0$ is caused by the second term containing $e^{-x/\epsilon}$ ceasing to be negligible. Hence we expect the boundary layer to occur when $x = O(\epsilon)$.

To solve problems like (6.5.1), we use the *method of matched asymptotic expansions*. We construct *two different* asymptotic expansions for the solution $y(x)$: one in the *outer region* where $x = O(1)$, and the other in the very narrow boundary layer near $x = 0$, also known as the *inner region*. Since these two expansions are approximating the *same function* $y(x)$, they must be self-consistent, and this allows them to be joined up by asymptotic *matching*.

6.5.2 Inner and outer expansions

To get the ideas clear, consider the example above where the exact solution (6.5.3) is known, and we want to find the inner and outer expansions. When $x = O(1)$, the second term in (6.5.3) is exponentially small, and thus

$$\begin{aligned} y(x;\epsilon) &\sim \frac{e^{-x}}{1-\epsilon} + \text{exp small} \\ &\sim e^{-x} + \epsilon e^{-x} + \cdots \quad \text{as } \epsilon \to 0, \end{aligned} \tag{6.5.4}$$

which reproduces the first two terms in the asymptotic expansion found in Exercise 6.14. This is the *outer expansion*, which applies when $x = O(1)$.

We can see from the exact solution (6.5.3) that the second term proportional to $e^{-x/\epsilon}$ stops being negligible when $x = O(\epsilon)$. We therefore examine the inner region by *rescaling* the independent variable. If we set $x = \epsilon X$ and $y(x;\epsilon) = Y(X;\epsilon)$, and now assume that $X = O(1)$ (corresponding to $x = O(\epsilon)$), then the exact solution (6.5.3) becomes

$$\begin{aligned} Y(X;\epsilon) &= \frac{e^{-\epsilon X} - e^{-X}}{1-\epsilon} \\ &\sim \left(1 - e^{-X}\right) + \epsilon\left(1 - X - e^{-X}\right) + \cdots \quad \text{as } \epsilon \to 0. \end{aligned} \tag{6.5.5}$$

This is the *inner expansion*, which is valid when $X = x/\epsilon = O(1)$.

6.5.3 Matching

In the previous section we showed how to create different asymptotic expansions of a single function which hold in different regions. Now we check that the two different approximations are self-consistent, in that they connect smoothly as x increases from $O(\epsilon)$ to $O(1)$. This method of joining two asymptotic expansions in different regions is called *matching*. For simplicity we restrict attention to only the leading-order terms outer and inner expansions (6.5.4) and (6.5.5), namely

$$y_0(x) = e^{-x}, \qquad\qquad Y_0(X) = 1 - e^{-X}, \tag{6.5.6}$$

with $X = x/\epsilon$. The two approximations are plotted in Figure 6.2. We see that the outer and inner solutions do indeed give good approximations to the exact solution (6.5.3) when $x = O(1)$ and when $x = O(\epsilon)$ respectively. The underlying principle of asymptotic matching is that both approximations should be valid in an intermediate *overlap region*.

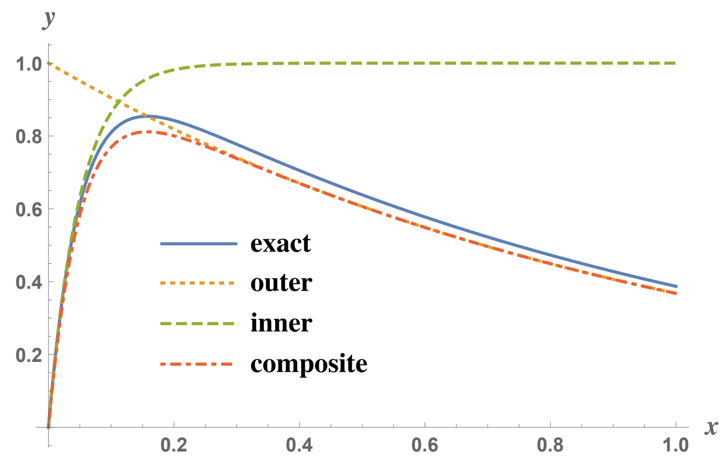

Figure 6.3: The exact expression (6.5.3) for $y(x;\epsilon)$, the leading-order inner and outer approximations (6.5.6), and the composite approximation (6.5.10), plotted with $\epsilon = 0.05$. plotted versus x with three different values of ϵ. The leading-order outer solution e^{-x} is plotted as a black dotted curve.

To examine such an overlap region, let us rescale $x = \delta\xi$ and $X = (\delta/\epsilon)\xi$, where δ is chosen to be intermediate between the inner and outer scalings for x, i.e. $\epsilon \ll \delta \ll 1$. The (6.5.6) becomes

$$y_0(\delta x) = \mathrm{e}^{-\delta x} \sim 1 + O(\delta) \quad \text{as } \delta \to 0, \tag{6.5.7a}$$

$$Y_0(\delta X/\epsilon) = 1 - \mathrm{e}^{-\delta X/\epsilon} \sim 1 + \text{exp small} \quad \text{as } \frac{\epsilon}{\delta} \to 0, \tag{6.5.7b}$$

and we see that the two approximations do agree and are both equal to 1 at lowest order in the overlap region.

A general statement of the leading-order matching principle demonstrated by (6.5.7) is

$$\lim_{x\to 0} y_0(x) = \lim_{X\to\infty} Y_0(X). \tag{6.5.8}$$

Loosely interpreted: the behaviour of the outer solution as we go *into* the boundary layer must equal the behaviour of the inner solution as we go *out* of the boundary later. More complicated versions of the matching principle (6.5.8) can be formulated to match inner and outer expansions up to arbitrary orders in ϵ, but we will only consider leading-order matching here.

Figure 6.3 demonstrates that the outer approximation works well when $x = O(1)$ but not when x is close to zero. Similarly, the inner approximation is good when x is small but not when $x = O(1)$. It is sometimes helpful to create a single function that gives a reasonable approximation for all values of x. Such a *composite* expansion can be constructed by forming

$$\text{composite expansion} = \text{inner expansion} + \text{outer expansion} - \text{common limit}, \tag{6.5.9}$$

where the common limit refers to components shared by the inner and outer approximations, which must subtracted to remove double-counting. At leading order, the common limit is given by $\lim_{x\to 0} y_0(x)$ or by $\lim_{X\to\infty} Y_0(X)$, and these two expressions are equal by the matching principle (6.5.8).

A composite expansion combining the inner and outer approximations (6.5.6) is given by

$$\begin{aligned} y_{\text{comp}}(x) &= \underbrace{y_0(x)}_{\text{outer}} + \underbrace{Y_0(X)}_{\text{inner}} - \underbrace{1}_{\text{common limit}} \\ &= \mathrm{e}^{-x} - \mathrm{e}^{-x/\epsilon}. \end{aligned} \tag{6.5.10}$$

Figure 6.3 verifies that (6.5.10) gives a good approximation to the exact solution (6.5.3) for all values of x.

6.5.4 Getting the expansion from the ODE

So far, we have constructed inner and outer approximations to a *known* solution (6.5.3). Now let us see whether we could have obtained the same approximations directly from the problem (6.5.1), if we did not have the exact solution to guide us. We have already seen that substitution of a naïve regular expansion of the form $y \sim y_0 + \epsilon y_1 + \cdots$ into (6.5.1) produces the outer approximation (6.5.4).

We note that (6.5.4) does not satisfy the boundary condition $y(0) = 0$, and we infer that the boundary condition can only be imposed if *the solution has a boundary layer at* $x = 0$. To examine this boundary layer, we have to rescale x: let us set $x = \delta X$ and $y(x) = Y(X)$ where $\delta \ll 1$ is to be determined. Then in terms of these inner variables, the problem (6.5.1) becomes

$$\frac{\epsilon}{\delta} Y'(X) + Y(X) = \mathrm{e}^{-\delta X}, \quad X > 0, \qquad Y(0) = 0. \tag{6.5.11}$$

We can balance all three terms in (6.5.11) by choosing $\delta = \epsilon$. We already know that the boundary layer thickness is of order ϵ from the exact solution (6.5.3), but here we determine the appropriate choice of δ directly by seeking a *dominant balance* in the ODE (6.5.11).

Once we have chosen $\delta = \epsilon$, the governing equation (6.5.11) in the inner region becomes

$$Y'(X) + Y(X) = \mathrm{e}^{-\epsilon X} \sim 1 - \epsilon X + \cdots. \tag{6.5.12}$$

Now we can seek an inner expansion of the usual form $Y \sim Y_0 + \epsilon Y_1 + \cdots$ and solve for each term successively. At leading order, we get

$$Y_0'(X) + Y_0(X) = 1, \qquad Y_0(0) = 0, \tag{6.5.13}$$

whose solution is easily found to be $Y_0(X) = 1 - \mathrm{e}^{-X}$, in agreement with (6.5.5). Thus we have successfully found the leading-order inner and outer approximations directly from the ODE and boundary conditions.

Before proceeding to apply the same ideas to more general BVPs, we note some general ideas that this simple example has illustrated.

(i) The boundary layer in the solution to (6.5.1) occurs because the small parameter ϵ multiplies the highest derivative in the ODE. When $x = O(1)$, we have

$$\underbrace{\epsilon y'(x)}_{\text{small}} + \underbrace{y(x) - \mathrm{e}^{-x}}_{\text{balance}} = 0 \tag{6.5.14}$$

and thus, in the limit as $\epsilon \to 0$, the *order* of the ODE is reduced, and we are no longer able to impose the boundary condition.

(ii) However, when there is a boundary layer, the derivative $y'(x)$ becomes very big (see e.g. Figure 6.2), such that the first term in (6.5.14) is no longer negligible at leading order.

(iii) This magnification of the gradient is represented by the change to the local variable $X = x/\epsilon$; by the chain rule we get $y'(x) = \epsilon^{-1}Y'(X)$.

(iv) The correct boundary layer scaling for x is found by seeking a dominant balance in the ODE; in particular, we want to bring the highest derivative back into the problem so that we are able to impose the boundary condition.

(v) The solutions of the inner and outer problems give us two alternative approximations for $y(x;\epsilon)$ — one that holds when $x = O(1)$ and one that holds when $x = O(\epsilon)$.

(vi) The leading-order inner and outer approximations can be reconciled by using the matching condition (6.5.8): the limit of the outer solution as we go into the boundary layer must equal the limit of the inner solution as we go out of the boundary layer.

In general, we can expect boundary layers (or something even worse) to occur whenever the small parameter ϵ multiplies the highest derivative in an ODE. The situation is analogous to Example 6.9, where we had to solve a quadratic equation with ϵ multiplying x^2. In both cases, if we set $\epsilon = 0$, the degree of the problem is reduced, and we do not obtain the full family of solutions. In both cases, the difficulty is resolved by rescaling x to get a dominant balance in the equation. In general, problems where setting ϵ to zero reduces the degree of the problem are called *singular perturbation* problems.

6.6 Boundary layers in BVPs

6.6.1 A simple example

In Example 6.14, we were unable to impose the boundary condition $y(0) = 0$ on the outer solution, and we deduced that there must be a boundary layer at $x = 0$. Once we found the inner and outer solutions, the matching condition (6.5.8) was satisfied identically: we could use it to verify that the inner and outer solutions are self-consistent, but it did not give us any further information about the solution. For higher-order BVPs, the situation is less clear. The location of any boundary layers may not be obvious in advance, and in general we will need to match the inner and outer approximations to determine the solution uniquely. We will illustrate the issues involved by solving a simple example.

Example 6.15. Find the leading-order solution of the BVP

$$\epsilon y''(x) + y'(x) = 1, \quad 0 < x < 1, \qquad y(0) = y(1) = 0 \tag{6.6.1}$$

in the limit $\epsilon \to 0$.

It is easy to solve (6.6.1) *exactly, but let us try to proceed using asymptotic expansions without assuming that we have the exact solution to hand.*

Outer solution *We try for a regular expansion with $y \sim y_0 + \epsilon y_1 + \cdots$ and obtain at leading order*

$$y_0'(x) = 1 \quad \Rightarrow \quad y_0(x) = x + A, \tag{6.6.2}$$

where A is an integration constant. Since the limit $\epsilon \to 0$ has reduced (6.6.1) from a second-order to a first-order ODE, we are unable to impose both of the boundary conditions. We deduce that there is a boundary layer somewhere, but where?

Let us assume for the moment that the boundary layer is at $x = 0$. This means that we can apply the boundary condition $y(1) = 0$ directly to the outer solution (6.6.2) and thus obtain

$$y_0(x) = x - 1. \tag{6.6.3}$$

Then the outer solution does not satisfy the boundary condition $y(0) = 0$, and we hope to resolve this by examining a boundary layer at $x = 0$.

Boundary layer *We find the size of the boundary layer by scaling $x = \delta X$ and $y(x) = Y(X)$, where $\delta \ll 1$ is to be determined. Putting this change of independent variables into the problem (6.6.1), we get*

$$\frac{\epsilon}{\delta^2} Y''(X) + \frac{1}{\delta} Y'(X) = 1. \tag{6.6.4}$$

Now we choose δ to achieve a dominant balance, in particular one that makes the highest derivative term no longer negligible. In this case this we achieve this by balancing the first two terms and thus taking $\delta = \epsilon$, so the ODE (6.6.4) becomes

$$Y''(X) + Y'(X) = \epsilon. \tag{6.6.5}$$

Now we can assume a simple expansion for the inner solution with $Y(X) \sim Y_0(X) + \epsilon Y_1(X) + \cdots$. At leading order we get

$$Y_0''(X) + Y_0'(X) = 0, \tag{6.6.6}$$

along with the boundary condition $Y_0(0) = 0$ (coming from the boundary condition for y at $x = 0$). The leading-order solution of the inner problem is thus given by

$$Y_0(X) = B\left(1 - \mathrm{e}^{-X}\right), \tag{6.6.7}$$

where B is an integration constant. Here we cannot solve for B, and therefore cannot determine the inner solution uniquely, using only the information in the boundary layer. To proceed, we must ensure that the inner and outer solutions match.

Matching *Now we impose the matching principle (6.5.8). In this case, the inner limit of the outer solution is $\lim_{x\to 0} y_0(x) = -1$, and the outer limit of the inner solution is $\lim_{X\to\infty} Y_0(X) = B$. The matching principle tells us that these must be equal, and hence $B = -1$ and the leading-order inner solution is given by*

$$Y_0(X) = -1 + \mathrm{e}^{-X}. \tag{6.6.8}$$

We can construct a composite expansion by combining (6.6.3) and (6.6.8), noting that the common limit here is equal to -1, to get

$$y_{\text{comp}}(x) = y_0(x) + Y_0(X) - (-1) = x - 1 + \mathrm{e}^{-x/\epsilon}, \tag{6.6.9}$$

which is a very good approximation of the exact solution of (6.6.1), namely

$$y(x) = x - \frac{1 - \mathrm{e}^{-x/\epsilon}}{1 - \mathrm{e}^{-1/\epsilon}}. \tag{6.6.10}$$

6.6.2 Locating the boundary layer

In Example 6.15, to get the leading-order solution, we assumed that the boundary layer is at $x = 0$, and therefore applied the boundary condition at $x = 1$ directly to the outer solution. The resulting leading-order approximation is in good agreement with the exact solution, but how could we have known in advance where to look for a boundary layer without having the exact solution to guide us?

Well, suppose that we had instead assumed the boundary layer to be at $x = 1$. We could attempt to analyse such a layer by using a local variable ξ such that $x = 1 - \delta\xi$ and $y(x) = \eta(\xi)$, with $\delta \ll 1$ to be determined. (It is not necessary to include the minus sign in the definition of ξ, but doing so means that we are dealing with $\xi > 0$ rather than $\xi < 0$.) Then equation (6.6.1) is transformed to

$$\frac{\epsilon}{\delta^2}\,\eta''(\xi) - \frac{1}{\delta}\,\eta'(\xi) = 1, \tag{6.6.11}$$

and a dominant balance between the first two terms is again achieved by choosing $\delta = \epsilon$. The leading-order problem in the inner region is thus

$$\eta_0''(\xi) - \eta_0'(\xi) = 0, \quad \xi > 0, \qquad\qquad \eta_0(0) = 0, \tag{6.6.12}$$

whose general solution is

$$\eta_0(\xi) = A\left(\mathrm{e}^{\xi} - 1\right), \tag{6.6.13}$$

where A is an integration constant. The problem is that the proposed inner solution (6.6.13) *grows exponentially* as ξ tends to infinity, and it is therefore impossible to match this solution to the solution in the outer region.

Note: In the above analysis, we assume that $0 < \epsilon \ll 1$. If $\epsilon = -|\epsilon|$ is negative, then the boundary layer *is* at $x = 1$, and the analysis in §6.6.1 needs to be redone.

There is a general principle for locating the boundary layers in simple two-point boundary-value problems like (6.6.1). Consider the general ODE

$$\epsilon y''(x) + P_1(x)y'(x) + P_0(x)y(x) = R(x), \quad a < x < b, \tag{6.6.14}$$

with boundary conditions given at $x = a$ and $x = b$. Assume that the coefficients P_0, P_1 and R are smooth and bounded, and that $P_1(x)$ *is non-zero* for $x \in [a, b]$.

The leading-order outer solution is found via a regular asymptotic expansion of the form $y \sim y_0 + \epsilon y_1 + \cdots$, which leads to

$$y_0'(x) + \frac{P_0(x)}{P_1(x)}\,y_0(x) = \frac{R(x)}{P_1(x)}. \tag{6.6.15}$$

This can be solved without difficulty on $[a, b]$ because of our assumptions about P_0, P_1 and R. However, because (6.6.15) is just a first-order ODE, we will be unable to impose both boundary conditions: there must be a boundary layer at one end of the domain, but which end?

Suppose we look for a boundary layer at $x = a$, via the re-scaling $x = a + \delta X$ and $y(x) = Y(X)$. It is clear that a dominant balance between the first two terms in (6.6.14) is achieved when $\delta = \epsilon$, and the leading-order inner equation is then

$$Y_0''(X) + P_1(a)Y_0'(X) = 0, \quad X > 0. \tag{6.6.16}$$

This has solutions of the form $Y_0(X) = A + B\mathrm{e}^{-P_1(a)X}$, and we can match with the outer only if the inner solution has a *decaying* exponential, i.e. if $P_1(a) > 0$.

Similarly, we can look for a boundary layer at $x = b$ with the scaling $x = b - \epsilon\xi$ and $y(x) = \eta(\xi)$, and get to leading order

$$\eta_0''(\xi) - P_1(b)\eta_0'(\xi) = 0, \quad \xi > 0. \tag{6.6.17}$$

Now the inner solution $\eta_0(\xi) = A + B\mathrm{e}^{P_1(b)\xi}$ can match with the outer only if $P_1(b) < 0$. Given our assumption that P_1 does not change sign, we conclude that:

- the boundary layer is at the *left-hand* boundary (i.e. $x = a$) if $P_1(x) > 0$, or
- at the *right-hand* boundary (i.e. $x = b$) if $P_1(x) < 0$.

One can imagine that more complicated behaviour is possible if $P_1(x)$ *does* change sign. The solution may have *two* boundary layers — one at each end of the domain — or an *internal* boundary layer somewhere in $a < x < b$ (and even more complicated structures are possible: see below).

6.7 More general perturbation methods for ODEs

6.7.1 Introduction

We have seen some examples of asymptotic methods applied to simple algebraic equations and ODE problems. More generally, ODEs containing small parameters can exhibit much more complicated behaviour than we have seen so far, and a range of asymptotic techniques have been developed to deal with them, which can be studied in more detail in C5.5 Perturbation methods. Here we give a brief (non-examinable) survey of some of the possible generalisations of the theory that has been developed so far.

6.7.2 Multiple or interior boundary layers

We argued in §6.6.2 that, in a second-order singular BVP, the location of the boundary layer can be predicted from the sign of the coefficient of the first derivative of y. But what happens if that coefficient changes sign somewhere in the domain? Here is a (relatively) simple example that illustrates what kind of behaviour can happen.

Example 6.16. Find the leading-order solution to the ODE

$$\epsilon y''(x) + y(x)y'(x) - y(x) = 0, \quad 0 < x < 1, \tag{6.7.1}$$

in the limit $\epsilon \to 0$, subject to each of the following sets of boundary conditions:

$$y(0) = 1, \qquad y(1) = 3; \tag{6.7.2a}$$
$$y(0) = -3/4, \qquad y(1) = 5/4; \tag{6.7.2b}$$
$$y(0) = 5/4, \qquad y(1) = -3/4. \tag{6.7.2c}$$

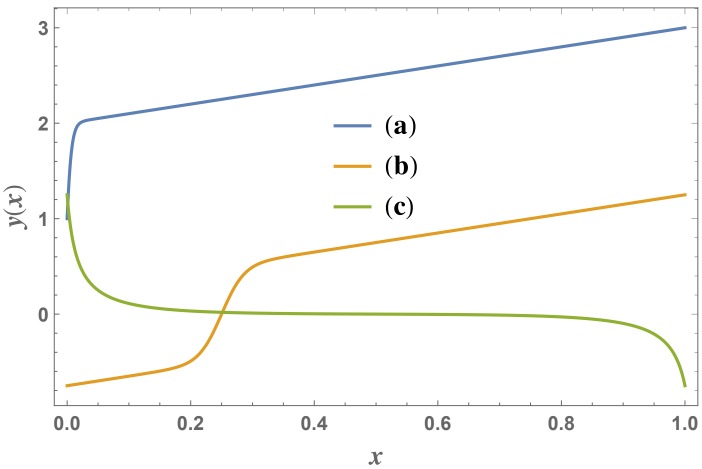

Figure 6.4: Solutions of the ODE (6.7.1) satisfying each of the boundary conditions (6.7.2), computed with $\epsilon = 0.01$.

In case (6.7.2a)*, the coefficient of y' in* (6.7.1) *is y, which is positive at both ends of the domain. The argument used in §6.6.2 works: there is a boundary layer only at $x = 0$. The leading-order inner and outer solutions may be found and matched in the usual way (with boundary layer thickness ϵ).*

In case (6.7.2b)*, the coefficient of y' in* (6.7.1) changes sign*, and it appears that a boundary layer is not allowed at either end of the domain. In this case, there is an* internal *boundary layer, at $x = x_*$ say, somewhere between $x = 0$ and $x = 1$. To solve the problem, we have to solve* two *outer problems: one in $0 < x < x_*$ and one in $x_* < x < 1$, and also solve for the boundary layer at $x = x_*$. By matching all three regions together, one can determine the location of the interior boundary layer (namely $x_* = 1/4$).*

Case (6.7.2c) *is even worse. In this case the signs of the coefficient of y' in* (6.7.1) *suggest that there might be a boundary layer at both ends of the domain. Indeed this turns out to be true, but the structure in this case is more complicated. The leading-order outer solution is given by $y_0(x) = 0$ (i.e. the other root of the leading-order outer equation $y_0(y_0' - 1) = 0$). The boundary layer at $x = 0$ has thickness ϵ again, but the inner solution in the boundary layer does not match directly with the outer solution. Instead, there is a further intermediate region in which $x = O\left(\epsilon^{1/2}\right)$ and $y = O\left(\epsilon^{1/2}\right)$. This is a so-called "triple deck" structure with one boundary layer nested inside another one. The boundary layer at $x = 1$ has an analogous structure.*

Numerically computed solutions to (6.7.1) *with $\epsilon = 0.01$ satisfying each of the boundary conditions in* (6.7.2) *are plotted in* Figure 6.4*. The structure of each solution is exactly as predicted: in case* (a) *there is just a boundary layer at $x = 0$; in case* (b) *there is an internal boundary layer close to $x = 1/4$; and in case* (c) *there is a boundary layer at both ends of the domain.*

Example 6.16 illustrates several issues that can arise in more complicated boundary layer problems. First: it may not be clear in advance where to look for boundary layers. Second: in general, the boundary layer analysis may require us to rescale the dependent variable y as well as the independent variable x. Finally: in the intermediate region encountered in Case (6.7.2c), we end up having to solve the full ODE, with no simplification (Try rescaling the ODE (6.7.1) with $x = \epsilon^{1/2}\xi$ and $y(x) = \epsilon^{1/2}\eta(\xi)$).

6.7.3 Slowly varying oscillations

In Example 6.12, we analysed small oscillations of a pendulum and found that we get spurious "secular" terms in the solution if we try a naïve regular asymptotic expansion. The origin of these terms can be understood by considering a very simple example.

Example 6.17. Solve the IVP

$$y''(x) + (1+\epsilon)y(x) = 0, \quad x > 0, \qquad y(0) = 1, \qquad y'(0) = 0. \tag{6.7.3}$$

The exact solution is

$$y(x) = \cos\left(x\sqrt{1+\epsilon}\right), \tag{6.7.4}$$

but if we try to expand this solution for small ϵ*, we get*

$$y(x) \sim \cos x - \frac{\epsilon}{2} x \sin x + \cdots. \tag{6.7.5}$$

Thus a secular term has appeared in the expansion, meaning that the expansion stops being valid when $x = O(1/\epsilon)$*. The fact that the exact solution* (6.7.4) *is a periodic function of* x *has become lost in our particular choice of asymptotic expansion.*

The difficulty encountered in Example 6.17 can be fixed relatively easily using the *Poincaré–Lindstedt method*. Here we know that we are seeking periodic solutions, but with a period that is a function of ϵ. The trick is to make the substitution

$$X = \omega x, \tag{6.7.6}$$

where the frequency ω is not known in advance, but is chosen to make the solution 2π-periodic as a function of X.

With $y(x) = Y(X)$, the problem (6.7.3) is transformed to

$$\omega^2 Y''(X) + (1+\epsilon)Y(X) = 0, \quad X > 0, \qquad Y(0) = 1, \qquad Y'(0) = 0. \tag{6.7.7}$$

Now we expand *both* Y *and* ω in powers of ϵ:

$$Y(X) \sim Y_0(X) + \epsilon Y_1(X) + \cdots, \qquad \omega \sim 1 + \epsilon\omega_1 + \cdots, \tag{6.7.8}$$

where we have anticipated that the leading-order frequency of oscillations is equal to 1.

At $O(1)$, we get

$$Y_0''(X) + Y_0(X) = 0, \quad X > 0, \qquad Y_0(0) = 1, \qquad Y_0'(0) = 0, \tag{6.7.9}$$

whose solution is

$$Y_0(X) = \cos X. \tag{6.7.10}$$

At $O(\epsilon)$, we find that $Y_1(X)$ satisfies the ODE

$$Y_1''(X) + Y_1(X) = -2\omega_1 Y_0''(X) - Y_0(X) = (2\omega_1 - 1)\cos X, \tag{6.7.11}$$

along with the initial conditions $Y_1(0) = Y_1'(0) = 0$, Now we insist that $Y_1(X)$ should be a 2π-periodic function of X, which means that it cannot contain any secular terms like $X \sin X$. We must therefore eliminate the "resonant" term proportional to $\cos X$ from the right-hand side of (6.7.11) by choosing $\omega_1 = 1/2$. Thus the oscillation frequency is given by an asymptotic expansion of the form

$$\omega \sim 1 + \frac{\epsilon}{2} + \cdots \quad \text{as } \epsilon \to 0, \tag{6.7.12}$$

which indeed agrees with the exact frequency $\omega = \sqrt{1+\epsilon}$ from equation (6.7.4).

The same method works for the problem of small oscillations of a pendulum from Example 6.12. Again the secular terms in the expansion can be suppressed and one can determine an asymptotic expansion for the frequency of the form $\omega \sim 1 - \epsilon^2/16 + O\left(\epsilon^4\right)$. The Poincaré–Lindstedt method is a simplified version of the more general *method of multiple scales*, which can describe oscillations that are not precisely periodic but instead vary slowly with x.

6.7.4 Fast oscillations

When our small parameter ϵ multiplies the highest derivative in an ODE, it does not always lead to the formation of boundary layers: it is also possible for the solution to exhibit rapid oscillations instead, as the following simple example shows

Example 6.18. Solve the BVP

$$\epsilon^2 y''(x) + y(x) = 0, \qquad y(0) = 1, \qquad y(1) = 0. \tag{6.7.13}$$

Note that, from the Fredholm Alternative, *we expect there to be problems whenever* $\epsilon = 1/\left(n^2\pi^2\right)$ *where* n *is an integer, but let's ignore that for the moment.*

If we try to proceed in the usual way by seeking the solution of (6.7.13) *as an asymptotic expansion in powers of* ϵ*, we just get* $y(x) \sim 0$*, to all algebraic orders in* ϵ*. Thus it appears to be impossible to impose the boundary conditions, and we might guess that there is a boundary layer at* $x = 0$*. But the inner rescaling* $x = \epsilon X$ *doesn't help, because the inner equation just gives oscillatory solutions which cannot match with the outer.*

One way to tackle problems like (6.7.13) is to use the *WKBJ method*. We seek the solution in the form

$$y(x) = A(x)\mathrm{e}^{\mathrm{i}u(x)/\epsilon}, \tag{6.7.14}$$

where both the *phase* $u(x)$ and the *amplitude* $A(x)$ are to be determined. By plugging the ansatz (6.7.14) into the ODE (6.7.13), we obtain

$$A(x)\left[1 - u'(x)^2\right] + \mathrm{i}\epsilon\left[2A'(x)u'(x) + A(x)u''(x)\right] + \epsilon^2 A''(x) = 0. \tag{6.7.15}$$

At leading order we get the *eikonal equation* $u'(x)^2 = 1$, and we deduce that the phase if simply given by $u(x) = \pm x$ (plus an irrelevant constant). We can then write the amplitude as a regular asymptotic expansion $A(x) \sim A_0(x) + \epsilon A_1(x) + \cdots$. In this simple problem, we just get $A'(x) = 0$, at all orders in ϵ, and indeed the ODE is solved exactly by $y(x) = A\mathrm{e}^{\pm\mathrm{i}x/\epsilon}$, with $A =$ constant. The general solution is then a linear combination of the form

$$y(x) = C_1\mathrm{e}^{\mathrm{i}x/\epsilon} + C_2\mathrm{e}^{-\mathrm{i}x/\epsilon}, \tag{6.7.16}$$

and the arbitrary constants can be determined from the boundary conditions.

Here is a slightly less trivial example, where we determine the asymptotic behavour of the zeroth order Bessel functions as the argument tends to infinity.

Example 6.19. Find the asymptotic behaviour of solutions to Bessel's equation of order zero:

$$y''(x) + \frac{1}{x}y'(x) + y(x) = 0, \tag{6.7.17}$$

in the limit as $x \to \infty$.

We can consider the behaviour for large x *by making the rescaling* $x = X/\epsilon$ *and* $y(x) = Y(X)$*, where* $\epsilon \ll 1$ *and* $X = O(1)$*. Then* (6.7.17) *is transformed to*

$$\epsilon^2 Y''(X) + \frac{\epsilon^2}{X}Y'(X) + Y(X) = 0. \tag{6.7.18}$$

Now we apply the WKBJ ansatz by writing $Y(X) = A(X)\mathrm{e}^{\mathrm{i}u(X)/\epsilon}$*, and* (6.7.18) *is transformed to*

$$\left[1 - u'(X)^2\right] + \mathrm{i}\epsilon\left[\left(\frac{2A'(X)}{A(X)} + \frac{1}{X}\right)u'(X) + u''(X)\right] + \epsilon^2\left[\frac{A''(X)}{A(X)} + \frac{A'(X)}{XA(X)}\right] = 0. \tag{6.7.19}$$

In this example, we get the same eikonal equation for $u(X)$ *as above, with solution* $u(x) = \pm X$*, and we are then left to solve*

$$\pm\left[\frac{2A'(X)}{A(X)} + \frac{1}{X}\right] - \mathrm{i}\epsilon\left[\frac{A''(X)}{A(X)} + \frac{A'(X)}{XA(X)}\right] = 0. \tag{6.7.20}$$

The leading-order amplitude therefore satisfies

$$\frac{A_0'(X)}{A_0(X)} = -\frac{1}{2X}, \tag{6.7.21}$$

whose solution is $A_0(X) = const/X^{1/2}$*. Thus solutions to* (6.7.18) *take the form*

$$Y(X) \sim \frac{C_1 \mathrm{e}^{\mathrm{i}X/\epsilon} + C_2 \mathrm{e}^{-\mathrm{i}X/\epsilon}}{\sqrt{X}} \quad \text{as } \epsilon \to 0. \tag{6.7.22}$$

In terms of the unscaled variable x*, we can write*

$$y(x) \sim \frac{c_1}{\sqrt{x}}\sin(x) + \frac{c_2}{\sqrt{x}}\cos(x) \quad \text{as } x \to \infty, \tag{6.7.23}$$

for some constants c_1 *and* c_2*.*

(The standard Bessel functions of the first and second kind *are normalised such that*

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right), \qquad Y_0(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4}\right) \tag{6.7.24}$$

as $x \to \infty$*.)*