Part A: Differential Equations II

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Contents

Health warning:

The following lecture notes are meant as a rough guide to the lectures. They are not meant to replace the lectures. You should expect that some material in these notes will not be covered in class and that extra material will be covered during the lectures (especially examples and applications). Nevertheless, I will try to follow the notation and the overall structure of the notes as much as possible. Also, these notes may be updated during the course of the term. In particular, please alert me if you catch any typos or errors. I will notify you if I upload an updated version.

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1   Introduction

In this course, we will develop various techniques for solving differential equations. Our primary concern will be finding ways to solve and understand inhomogeneous linear boundary value problems (BVPs), that is an ordinary differential equation (ODE)

$$Ly(x) = f(x), \ \ a < x < b \tag{1}$$

where L is a *linear differential operator* of the form

$$Ly(x) \equiv a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y(x) + a_0 y(x) \tag{2}$$

and the function $f(x)$ on the right hand side (RHS) is a given *forcing function* in the system. The operator L is linear in the sense that

$$L(\alpha_1 y_1(x) + \alpha_2 y_2(x)) = \alpha_1 L y_1(x) + \alpha_2 L y_2(x)$$

for any constants α_i and functions $y_i(x)$.

Along with equation (1) we require n **boundary conditions**. For a linear system, this consists of n linear combinations of $y(x)$ and its derivatives up

to order $n-1$ evaluated at the boundary points $x = a$ and $x = b$. We will write these generically as

$$B_i y\Big|_{x=a,b} = \gamma_i, \quad i = 1, 2, \ldots n.$$

For instance, if $n = 2$, a 2nd order system, the boundary conditions would have the form

$$
\begin{aligned}
B_1 y &:= \alpha_1 y(a) + \alpha_2 y(b) + \beta_1 y'(a) + \beta_2 y'(b) \\
B_2 y &:= \tilde{\alpha}_1 y(a) + \tilde{\alpha}_2 y(b) + \tilde{\beta}_1 y'(a) + \tilde{\beta}_2 y'(b)
\end{aligned}
\tag{3}
$$

for constants $\alpha_{1,2}, \beta_{1,2}$.

Note: it is useful to distinguish between the differential system, comprised of the operator L and the boundary terms B_i; and the *data*: the function $f(x)$ and the boundary values $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$.

The key elements in our analysis are:

1. **BVP.** A boundary value problem is fundamentally different from an initial value problem (IVP), which formed the crux of the Differential Equations 1 course. In an IVP, the extra conditions needed to complete the system are all defined at a single point. In IVPs, the independent variable is often time t, and hence data is given at the initial time $t = 0$.

   In a boundary value problem, data is given at multiple points[1]. For BVPs, the independent variable is often a spatial dimension, and data is provided about the solution at the ends of the spatial domain.

2. **Inhomogeneity.** An important element in our development is the presence of the arbitrary function $f(x)$ in (1). In a physical system, the forcing function represents some action of the outside world. The presence of such an inhomogeneity greatly affects solution behaviour and complicates analysis.

3. **Linearity.** In general, an inhomogeneous BVP can be quite challenging and often intractable. In nearly of our developments, we will exploit linearity: through much of the course will use the linearity of the operator and boundary conditions, and at the end we will consider asymptotic approaches, which can be a means of finding linear approximations to nonlinear systems.

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[1]Note: some textbooks will define a BVP simply as a differential equation plus extra conditions, in which case an IVP is a subset of a BVP.

Some questions we will consider:

1. How do we solve the system for an arbitrary function $f(x)$?

2. Is there always a solution? If so, is it unique?

3. What is the effect of the boundary conditions?

4. Can we solve if the $a_k = a_k(x)$ are functions of x?

5. Can we exploit the presence of a small parameter in the system?

1.1 Homogeneous vs inhomogeneous

Consider the second order operator

$$Ly \equiv P_2(x)y''(x) + P_1(x)y'(x) + P_0(x)y(x), \quad a \le x \le b$$

(and ignore boundary conditions for now).
We distinguish between two equation types:

$$Ly(x) = 0 : H, \text{ homogeneous}$$
$$Ly(x) = f(x) : N, \text{ inhomogeneous}$$

for some given f.

1.2 Solutions

The following properties of solutions of H and N are easily established:

(i) the solutions of H form a vector space (since if $Ly_1 = 0 = Ly_2$ then $L(\alpha y_1 + \beta y_2) = 0$).

(ii) if y_1 and y_2 satisfy N then $y_1 - y_2$ satisfies H, so that the general solution of N may be written

$$y = \underbrace{y_{PI}}_{\text{any solution of } N} + \underbrace{y_{CF}}_{\text{general solution of } H}$$

where y_{PI} is called the *particular integral* and y_{CF} the *complementary function*.

1.3 Linear independence of functions

A pair of functions $y_1(x)$, $y_2(x)$ is linearly independent if the only linear combination that vanishes (identically):

$$c_1 y_1(x) + c_2 y_2(x) \equiv 0$$

has $c_1 = c_2 = 0$. They are linearly *dependent* if such c_i, not both zero, can be found. If the y_i are also differentiable then this would entail

$$c_1 y_1' + c_2 y_2' = 0,$$

i.e.

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

so that the determinant of the matrix is zero.

Define **the Wronskian** of a pair of function to be this determinant:

$$W(x; y_1, y_2) = y_1 y_2' - y_2 y_1'. \tag{4}$$

From what we have just seen:

Proposition If two functions are linearly dependent then their Wronskian vanishes.

The converse to this isn't obvious; consider the following pair of (once) differentiable functions:

$$y_1 = \begin{cases} 0 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

$$y_2 = \begin{cases} x^2 & x < 0 \\ 0 & x \geq 0 \end{cases}$$

then $W = 0$. However, if $c_1 y_1 + c_2 y_2 = 0$ then evaluation for positive and negative x shows that $c_1 = c_2 = 0$, so that these functions are in fact linearly independent. To establish a partial converse note the following:

Proposition If y_1 and y_2 satisfy H, i.e.

$$P_2 y_i'' + P_1 y_i' + P_0 y_i = 0,$$

then their Wronskian W satisfies

$$\frac{W'}{W} = -\frac{P_1}{P_2}.$$

The proof is an easy exercise. Solving for W, we get

$$W = const \times \exp\left[-\int^x \frac{P_1(t)}{P_2(t)}\, dt\right]. \tag{5}$$

In particular, **provided P_2 is nowhere zero**, if $W = 0$ at one point, then $W = 0$ at every point (since in this case the exponential can't vanish so this can only happen if the constant in front of (5) is zero).

1.4 A basis of solutions of H

We choose solutions y_1 and y_2 of H with

$$\begin{aligned} y_1(a) &= 1, \quad y_1{}'(a) = 0 \\ y_2(a) &= 0, \quad y_2{}'(a) = 1. \end{aligned}$$

By the work in DE's 1, these exist and are unique at least in a neighbourhood of $x = a$ provided $P_2(a) \neq 0$. Also their Wronskian $W(x)$ has $W(a) = 1$, so is nonzero in this neighbourhood of $x = a$, and they are linearly independent. Do they span the vector space of solutions? Suppose $y(x)$ is any other solution and set

$$Y(x) = y_1(x)y(a) + y_2(x)y'(a).$$

Then this is a solution with

$$Y(a) = y(a); \ Y'(a) = y'(a)$$

and so by uniqueness $Y(x) = y(x)$. Thus, $y(x)$ is a linear combination of y_1 and y_2 and hence they do span the vector space of solutions, i.e. they are a basis. We can conclude:

Proposition

(i) The dimension of the space of solutions of H is 2.

(ii) Any pair of solutions of H with $W \neq 0$ is a basis.[2]

Exercise: generalise everything done so far to n-th order linear ODEs.

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[2]Remark: Any linearly independent pair of solutions is also a basis, and so will be related to the basis of Section 1.4 by a nonsingular matrix. At a, their Wronskian will be the determinant of this matrix, and therefore nonzero. This gives our partial converse to the first Proposition: two solutions of a given fixed H are linearly dependent if and only if their Wronskian is zero. This fact can be proved directly as follows: suppose u and v are two solutions of H; if they are linearly dependent then we know already that their Wronskian is zero so now suppose for the converse that their Wronskian is zero; if u is the

1.5 Finding the general solution of the Homogeneous problem

For the 2nd order ODE

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0,$$

there are a few standard methods for finding the general solution.

Constant coefficients

If P_2, P_1 and P_0 are constants then guessing the solution $y = e^{mx}$ will result in the auxiliary equation for m and the general solution can then easily be found (care must be taken for cases where the roots, m, are complex or are repeated.)

Cauchy-Euler equation

If the coefficients are of the form $P_2(x) = \alpha x^2$, $P_1(x) = \beta x$, $P_0(x) = \gamma$, with α, β, γ constants, then a solution can be found with the guess $y = x^m$ (again taking care if the roots m are repeated or complex).

Reduction of order

If one solution, $y_1(x)$, is known then the general solution can be found by solving an ODE of reduced order. The method is to assume that the solution to the full ODE can be written as

$$y(x) = v(x)y_1(x).$$

Note we know that the function $v(x) = C$ is a possible answer but we seek something more general. Putting this into the ODE gives

$$P_2(vy_1'' + 2v'y_1' + v''y_1) + P_1(vy_1' + v'y_1) + P_0vy = 0 \ .$$

Using the fact that $y_1(x)$ satisfies the ODE, we are left with

$$P_2(2v'y_1' + v''y_1) + P_1v'y_1 = 0 \ .$$

This is a separable first order ODE for the function $z = v'(x)$:

$$P_2y_1z' + (2P_2y_1' + P_1y_1)z = 0 \ .$$

---

zero function then they are certainly linearly dependent so suppose that there is at least one value of x, say $x = a$, with $u(a) \neq 0$; pick μ so that $v(a) = \mu u(a)$ then

$$0 = W(a) = u(a)v'(a) - v(a)u'(a) = u(a)(v'(a) - \mu u'(a));$$

cancel $u(a)$ to conclude that $v'(a) = \mu u'(a)$; now define $y(x) = v(x) - \mu u(x)$, then $y(x)$ is a solution of H by linearity, while $y(a) = 0 = y'(a)$ by the choices made so far; thus by uniqueness of solution of H we conclude that $y(x) = 0$ and therefore u and v are linearly dependent QED.

Solving for $z(x)$ then gives the general solution

$$y = v(x)y_1(x).$$

Note all three methods can be used for higher order problems with similar properties.

1.6 Variation of parameters

We now know a good deal about the solutions of H. Much of this course is concerned with solving for N. 'Variation of parameters' is the first method we will encounter to do so. Recall the distinction:

$$Ly(x) \equiv P_2 y'' + P_1 y' + P_0 y = \begin{cases} 0 & : H \\ f & : N \end{cases},$$

and suppose that H is solved by $y = c_1 y_1(x) + c_2 y_2(x)$ with linearly independent y_1, y_2. We seek a solution of N of the form

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x), \tag{6}$$

i.e. we 'vary the parameters'. Using two functions, c_1 and c_2, to find one, $y(x)$, we expect to be able to impose another condition on the c_i.
First, differentiate (6) to find

$$y' = c_1 y_1' + c_2 y_2' + c_1' y_1 + c_2' y_2$$

and now *impose*

$$c_1' y_1 + c_2' y_2 = 0. \tag{7}$$

The justification for this is that it will simplify the equations for c_1 and c_2 so that we can find an explicit formula for the solution of N (otherwise we just get two 2nd order ODEs to solve and have gained nothing). Differentiating again,

$$y'' = c_1 y_1'' + c_2 y_2'' + c_1' y_1' + c_2' y_2'$$

so that

$$Ly = P_2(c_1 y_1'' + c_2 y_2'' + c_1' y_1' + c_2' y_2')$$
$$+ P_1(c_1 y_1' + c_2 y_2')$$
$$+ P_0(c_1 y_1 + c_2 y_2).$$

But, since the y_i satisfy H, this gives

$$Ly = P_2(c_1' y_1' + c_2' y_2') = f. \tag{8}$$

Solve ([7](#)) and ([8](#)) for c_1' to find

$$c_1' = -\frac{f y_2}{P_2 W}, \ W = y_1 y_2' - y_2 y_1'$$

and then

$$c_2' = \frac{f y_1}{P_2 W}.$$

We can integrate these to obtain

$$\left.\begin{array}{l} c_1(x) = -\displaystyle\int^x \frac{f(t) y_2(t)}{P_2(t) W(t)}\, dt \\[2mm] c_2(x) = \displaystyle\int^x \frac{f(t) y_1(t)}{P_2(t) W(t)}\, dt \end{array}\right\} \qquad (9)$$

$$y(x) = -\int^x \frac{f(t) y_2(t) y_1(x)}{P_2(t) W(t)}\, dt + \int^x \frac{f(t) y_1(t) y_2(x)}{P_2(t) W(t)}\, dt \qquad (10)$$

The freedom in the choice of lower limit in these integrals gives additive constants in c_i and so, by ([6](#)), adds a solution of H to this solution of N.

1.7 Fixing the boundary values

We now develop a method for solving BVP (N) with homogeneous boundary conditions for any nonhomogeneity $f(x)$. We consider the BVP

$$P_2 y'' + P_1 y' + P_0 y = f \qquad (11)$$

with boundary data

$$y(a) = 0 = y(b). \qquad (12)$$

We follow the Variation of Parameters recipe, but we choose y_1 so that $y_1(a) = 0$, and y_2 so that $y_2(b) = 0$. (We assume this can be done and return to this point later.) So

$$y(x) = c_1(x) y_1(x) + c_2(x) y_2(x)$$

with the c_i as in ([9](#)). Then the boundary data requires

$$y(a) = c_1(a) y_1(a) + c_2(a) y_2(a) = 0$$

$$y(b) = c_1(b) y_1(b) + c_2(b) y_2(b) = 0.$$

and, with the choices made for y_i, this requires that we take $c_2(a) = 0 = c_1(b)$. Imposing these conditions on (9) we therefore need to take

$$
\left.
\begin{aligned}
c_1(x) &= \int_x^b \frac{f(t)y_2(t)}{P_2(t)W(t)} \, dt, \\
c_2(x) &= \int_a^x \frac{f(t)y_1(t)}{P_2(t)W(t)} \, dt
\end{aligned}
\right\}
\tag{13}
$$

(note the change of limits). Thus we fix the c_i completely. Now the solution can be written as

$$
y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)
$$
$$
= \int_a^x \frac{f(t)y_1(t)y_2(x)}{P_2(t)W(t)} \, dt + \int_x^b \frac{f(t)y_2(t)y_1(x)}{P_2(t)W(t)} \, dt
$$

which we can write concisely as

$$
y(x) = \int_a^b g(x,t)f(t) \, dt
\tag{14}
$$

where

$$
g(x,t) =
\begin{cases}
\dfrac{y_1(t)y_2(x)}{P_2(t)W(t)} & a \le t \le x \le b \\[2mm]
\dfrac{y_2(t)y_1(x)}{P_2(t)W(t)} & a \le x \le t \le b
\end{cases}
\tag{15}
$$

We call $g(x,t)$ the *Green's function*, something we explore fully in Section 3.

1.8   An example

Consider the BVP

$$
y'' + y = f(x) \quad \text{for } 0 \le x \le \frac{\pi}{2}
$$

with data

$$
y(0) = 0 = y\left(\frac{\pi}{2}\right),
$$

and run through the method:

- Identify H as $y'' + y = 0$.

- Choose solution y_1 with $y_1(0) = 0$ so $y_1 = \sin x$ will do.

- Choose solution y_2 with $y_2\left(\frac{\pi}{2}\right) = 0$ so $y_2 = \cos x$ will do.

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- Calculate

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -1.$$

- Note $P_2 = 1$

and so by (51)

$$g(x,t) = \begin{cases} -\sin t \cos x & 0 \le t \le x \le \frac{\pi}{2} \\ -\cos t \sin x & 0 \le x \le t \le \frac{\pi}{2} \end{cases}$$

The solution of the BVP is then, by (50):

$$y(x) = \int_0^{\frac{\pi}{2}} g(x,t)f(t)\,dt,$$

with this g.

2 Eigenfunction methods

Our next approach to solving linear inhomogeneous BVP's is through an eigenfunction expansion. The idea is to exploit the linearity of the operator by constructing a solution as a *superposition* of a (generally infinite) set of functions $\{y_i(x)\}$. In particular, the y_i will be the functions satisfying

$$Ly_i(x) = \lambda_i y_i(x), \tag{16}$$

along with homogeneous boundary conditions. Here y_i is an *eigenfunction* with corresponding *eigenvalue* λ_i. This is analogous to the linear algebra eigenproblem

$$\mathbf{A}\vec{x}_i = \lambda_i \vec{x}_i \tag{17}$$

where $\mathbf{A}$ is a matrix and $\vec{x}_i$ an eigenvector with eigenvalue λ_i.

2.1 Sets of functions

Similar to a vector space, we can introduce a set of linearly independent basis functions $y_n(x), n = 1, 2, \ldots \infty$ such that any 'reasonable' function $f(x)$ can be written as a linear combination of these functions[3]:

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x). \tag{18}$$

---

[3]Note the difference between this statement and the discussion of Sec. 1.4: there we found a 2D basis of solutions to the homogeneous problem, here we mean a basis for any function, hence the basis becomes infinite dimensional.

You have encountered this idea before with Fourier Series, where the basis functions are sines and cosines; this is merely a generalisation. Hence it should be clear that we can have different sets of basis functions.

We also define the **inner product**

$$\langle u, v \rangle := \int_a^b u(x)\overline{v(x)} \, dx. \tag{19}$$

Here the overbar denotes complex conjugate. In this course, we will rarely be concerned with complex valued functions. If it is clear that we are dealing with real functions, we may drop the overbar for simplicity.

2.2 Adjoint

We also require the notion of the *adjoint* of an operator. For operator L with homogenous BC, the adjoint problem $(L^* \text{ BC*})$ is defined by the inner product relation

$$\langle Ly, w \rangle = \langle y, L^*w \rangle. \tag{20}$$

To determine the adjoint, one needs to move the derivatives of the operator from y to w, and define adjoint boundary conditions so that all boundary terms vanish.

Example

Let $Ly = \dfrac{d^2 y}{dx^2}$ with $a \leqslant x \leqslant b$, $y(a) = 0$ and $y'(b) - 3y(b) = 0$. We wish to find L^*w, such that

$$\int_a^b (w)(y'')dx = \int_a^b (y)(L^*w)dx$$

To do this, we need to shift the derivatives from y to w using integration by parts:

$$
\begin{aligned}
\int_a^b wy''dx &= wy'|_a^b - \int_a^b w'y'dx \\
&= wy' - w'y|_a^b + \int_a^b yw''dx
\end{aligned}
$$

The integral gives the formal part so:

$$L^*w = \frac{d^2 w}{dx^2}.$$

The inner product only includes integral terms, so *the boundary terms must vanish*, which will define boundary conditions on w, i.e. this defines BC*. Here, we require

$$w(b)y'(b) - w'(b)y(b) - w(a)y'(a) + w'(a)y(a) = 0.$$

Using the BC's $y'(b) = 3y(b)$ and $y(a) = 0$, gives:

$$0 = y(b)\Big(3w(b) - w'(b)\Big) - w(a)y'(a) + \underbrace{w'(a)y(a)}_{=0}$$

As these terms need to vanish for all values of $y(b)$ and $y'(a)$, we can infer two boundary conditions on w:

- $y(b)$: $3w(b) - w'(b) = 0$

- $y'(a)$: $w(a) = 0$

If $L = L^*$ and $BC = BC^*$ then the problem is *self-adjoint*. If $L = L^*$ but $BC \neq BC^*$ we still call the operator self-adjoint. (Some books use the terminology *formally self-adjoint* if $L = L^*$ and *fully self-adjoint* if both $L = L^*$ and $BC = BC^*$).

2.2.1 Eigenfunction Properties

The main idea in solving the BVP is to construct a solution as a linear combination of eigenfunctions. There are two fundamental properties of eigenfunctions that will be vital to this approach.

1. **Eigenfunctions of the adjoint problem have the same eigenvalues as the original problem**

   That is,
   $$Ly = \lambda y \Rightarrow \exists w \ni L^* w = \lambda w.$$

2. **Eigenfunctions corresponding to different eigenvalues are orthogonal**

   That is, if $Ly_j = \lambda_j y_j$ (so $L^* w_j = \lambda_j w_j$) and $Ly_k = \lambda_k y_k$ ($L^* w_k = \lambda_k w_k$), then for $\lambda_j \neq \lambda_k$, $\langle y_j, w_k \rangle = 0$.

Proof

$$\begin{aligned}
\lambda_j \langle y_j, w_k \rangle &= \langle \lambda_j y_j, w_k \rangle \\
&= \langle L y_j, w_k \rangle \\
&= \langle y_j, L^* w_k \rangle \\
&= \langle y_j, \lambda_k w_k \rangle \\
&= \lambda_k \langle y_j, w_k \rangle.
\end{aligned}$$

But $\lambda_j \neq \lambda_k$ so $\langle y_j, w_k \rangle = 0$. (The proof is exactly as for matrix problems.)

2.3 Inhomogeneous solution process

We are now in a position to outline the construction of solution to the BVP

$$Ly = f(x)$$

with linear, homogeneous boundary conditions, denoted $BC_1(a) = 0$, $BC_2(b) = 0$.

Step 1: Solve the eigenvalue problem

$$Ly = \lambda y, \quad BC_1(a) = 0, \ BC_2(b) = 0$$

to obtain the eigenvalue-eigenfunction pairs (λ_j, y_j).

Step 2: Solve the adjoint eigenvalue problem

$$L^* w = \lambda w, \quad BC_1^*(a) = 0, \ BC_2^*(b) = 0$$

to obtain (λ_j, w_j).

Step 3: Assume a solution to the full system $Ly = f(x)$ of the form

$$y = \sum_i c_i y_i(x).$$

To determine the coefficients c_i, start from $Ly = f$ and take an inner product with w_k:

$$\begin{aligned}
Ly &= f(x) \\
\Rightarrow \langle Ly, w_k \rangle &= \langle f, w_k \rangle \\
\Rightarrow \langle y, L^* w_k \rangle &= \langle f, w_k \rangle \\
\Rightarrow \langle y, \lambda_k w_k \rangle &= \langle f, w_k \rangle \\
\Rightarrow \lambda_k \langle \sum_i c_i y_i, w_k \rangle &= \langle f, w_k \rangle \\
\Rightarrow \lambda_k c_k \langle y_k, w_k \rangle &= \langle f, w_k \rangle
\end{aligned} \tag{21}$$

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We can solve the last equality for the c_k, and we are done! Note that in the last step we have used the orthogonality property $\langle y_j, w_k \rangle = 0, \; j \neq k$.

2.4 A note on boundary conditions

In the above construction we assumed homogeneous boundary conditions. In the general case of an inhomogeneous system with inhomogenous boundary conditions,

$$Ly = f(x)$$
$$B_i y = \gamma_i \tag{22}$$

a useful technique is to split the system in two, i.e. solve both

$$Ly_1 = f(x), \quad B_i y_1 = 0 \tag{23}$$

and

$$Ly_2 = 0, \quad B_i y_2 = \gamma_i. \tag{24}$$

Here, solving for $y_1(x)$ has the difficulty of the forcing function but with zero BC's while the other equation is homogeneous but has the non-zero BC's. Due to linearity, it is easy to see that $y(x) = y_1(x) + y_2(x)$ solves the full system (22).

This decomposition can always be performed[4] and since solving (24) tends to be an easier matter (for linear systems!), it is safe for us to primarily focus on the technique of solving the system (23), i.e. homogeneous boundary conditions.

For completeness it is worth noting that one *can* solve BVPs with inhomogeneous BC using an eigenfunction expansion and without needing a decomposition. The keys are:

1. The eigenfunctions are *always* determined using homogeneous boundary conditions. Thus, eigenfunctions won't change whether you "decompose" or not. The difference comes in:

2. In going from Line 2 to 3 of (21), care must be taken in the integration by parts, as boundary terms will generally still be present. (Can you see why?) These extra boundary terms then carry through to the formula for the c_k.

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[4]As we shall see in Section 4, it requires caution if there is a zero eigenvalue $\lambda = 0$.

2.5 Example

Let $y'' = f(x)$ with $0 \leqslant x \leqslant 1$, $y(0) = \alpha$ and $y(1) = \beta$. Then:

BC's Incorporated Solution Route

1. Solve $y'' = \lambda y$, with $y(0) = 0$ and $y(1) = 0$.

   We get $y_k(x) = \sin(k\pi x)$ and $\lambda_k = -k^2\pi^2$ with $k = 1, 2, 3, \ldots$.

   The problem is self-adjoint (show this as an exercise), so $w_k = y_k = \sin(k\pi x)$ and $w_k'' = \lambda_k w_k$.

2.

$$
\begin{aligned}
y'' &= f(x) \\
\int_0^1 w_k y'' \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \\
\Rightarrow (y'w_k - yw_k')|_0^1 + \int_0^1 w_k'' y \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \\
\Rightarrow (y'w_k - yw_k')|_0^1 + \lambda_k \int_0^1 w_k y \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \\
\Rightarrow (y'w_k - yw_k')|_0^1 + \lambda_k c_k \int_0^1 w_k y_k \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \\
\Rightarrow (y'w_k - yw_k')|_0^1 - k^2\pi^2 c_k \int_0^1 \sin^2(k\pi x) \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x
\end{aligned}
$$

3. Now $\int_0^1 \sin^2(k\pi x) \, dx = 1/2$, and $w_k = \sin(k\pi x)$, hence

$$
y'w_k - yw_k'|_0^1 = -k\pi \cos(k\pi)y(1) + k\pi \cos(0)y(0)
$$

$$
\Rightarrow -\beta k\pi(-1)^k + \alpha k\pi - \frac{1}{2}k^2\pi^2 c_k = \int_0^1 f(x) \sin k\pi x \mathrm{d}x
$$

   Solving for c_k gives us $y(x)$ as a Fourier series.

Decomposed Solution Route

1. Solve two systems separately:

$$y'' = f(x), \quad y(0) = y(1) = 0$$
$$u'' = 0, \quad u(0) = \alpha, \ u(1) = \beta$$

2. To solve for y, since BC=0 we can jump straight to the formula

$$c_k = \frac{\langle f, w_k \rangle}{\lambda_k \langle y_k, w_k \rangle} = -\frac{2 \int_0^1 f(x) \sin(k\pi x) \, dx}{k^2 \pi^2}.$$

3. The solution for u is easily obtained as

$$u = (\beta - \alpha)x + \alpha$$

4. The full solution is $y(x) + u(x)$.

Although they look different, both approaches give the same solution. Either way, we see that self-adjoint problems are great: they are less work since the w_k's are the same as the y_k's.

2.6 Connection with linear algebra

There are direct parallels between linear algebra and linear BVPs:

$$\underline{\text{Linear algebra}} \qquad \underline{\text{Linear BVP}}$$
$$\text{vector } \vec{v} \in \mathbb{R}^n \quad \longleftrightarrow \quad \text{function } y(x) \text{ for } a \leqslant x \leqslant b$$

$$\underbrace{\vec{v} \cdot \vec{w} = \sum_{k-1}^n v_k w_k}_{\text{dot product}} \quad \longleftrightarrow \quad \underbrace{\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\mathrm{d}x}_{\text{inner product}}$$

$$\underbrace{\| \vec{v} \|^2 = \vec{v} \cdot \vec{v} \geqslant 0}_{\text{norm}} \quad \longleftrightarrow \quad \underbrace{\| f \|^2 = \langle f, f \rangle \geqslant 0}_{\text{norm}}$$

$$\perp \text{ vector } \vec{v} \cdot \vec{w} = 0 \quad \longleftrightarrow \quad \text{orthogonal function } \langle f, g \rangle = 0$$

$$\text{Matrix } A \quad \longleftrightarrow \quad \text{Linear Differential Operator } L$$

Given a vector $\vec{v}$, then the product $A\vec{v}$ is a new vector. Similarly, given a function $y(x)$,

$$Ly = a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy$$

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evaluates to a new function on $a \leqslant x \leqslant b$.

In linear algebra, a common problem is to solve the equation

$$A\vec{v} = \vec{b}$$

for $\vec{v}$, given matrix A and vector $\vec{b}$. Compare that to our general task of solving $Ly = f$ for y, given operator L and RHS f.

Eigenvalue problems

Linear algebra		Linear BVP
$A\vec{v} = \lambda\vec{v}$	$\longleftrightarrow$	$Ly = \lambda y$

How many eigenvalues?

Linear algebra	Linear BVP
A is $n \times n$	L is order n
Solve $|A - \lambda I| = 0$	
$\Rightarrow n$ eigenvalues	∞ eigenvalues

Adjoint

Linear algebra	Linear BVP
$A \to A^T$	$L \to L^*$
	BC's $\to$ BC*'s

Self adjoint if $\quad A = A^T \qquad L = L^*, \ \ \text{BC=BC*}$

A self-adjoint matrix is called Hermitian. A self-adjoint linear differential operator is also referred to as Hermitian. We next look at a particular class of Hermitian operator – Sturm-Liouville operators – that occurs quite commonly and has very useful properties.

2.7 Sturm-Liouville theory

Sturm–Liouville (SL) theory of second order concerns self-adjoint operators of the form:

$$Ly = \lambda r(x)y$$

where $r(x) \geqslant 0$ is a *weighting function*, and the operator L is of the form

$$Ly = -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y, \quad a \leq x \leq b \tag{25}$$

The functions p, q, and r are all assumed to be real. It is easy to check that the operator is formally self-adjoint. It is fully self-adjoint if the boundary conditions take the separated form

$$\begin{aligned}
\alpha_1 y(a) + \alpha_2 y'(a) &= 0 \\
\alpha_3 y(b) + \alpha_4 y'(b) &= 0.
\end{aligned}$$

Observe also that if $p(a) = p(b) = 0$, then $\langle Ly, w \rangle = \langle y, Lw \rangle$ irrespective of boundary conditions. This defines the so-called natural interval $[a, b]$ for the problem.

2.7.1 Inhomogeneous SL problems

Since a SL operator is self-adjoint, the eigenfunction expansion process is quite straightforward. Consider

$$Ly = f(x)$$

with homogeneous BC's. The system can be solved with an eigenfunction expansion in the same manner as in Section 2.3:

$$\begin{aligned}
Ly &= f(x) \\
\Rightarrow \langle Ly, y_k \rangle &= \langle f, y_k \rangle \\
\Rightarrow \langle y, Ly_k \rangle &= \langle f, y_k \rangle \quad \text{(since } L^* = L, w_k = y_k) \tag{26} \\
\Rightarrow \langle y, \lambda_k r y_k \rangle &= \langle f, y_k \rangle \\
\Rightarrow \lambda_k c_k \langle y_k, r y_k \rangle &= \langle f, y_k \rangle.
\end{aligned}$$

Thus we obtain the formula

$$c_k = \frac{\langle f, y_k \rangle}{\lambda_k \langle y_k, r y_k \rangle} \tag{27}$$

and the full solution is given by

$$y = \sum_k c_k y_k.$$

2.7.2 Transforming an operator to SL form

Many problems encountered in physical systems are Sturm-Liouville. In fact, though, any operator

$$Ly \equiv a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x)$$

with $a_2(x) \neq 0$ in the interval can be converted to a SL operator.
To *transform* to a self-adjoint SL operator, multiply by an integrating factor function $\mu(x)$:

$$\mu a_2(x)y''(x) + \mu a_1 y'(x) + \mu a_0 y$$

We then choose μ so that the first and second derivatives collapse, i.e. so it can be expressed in the form

$$-\frac{d}{dx}(py') + qy$$

Suppose we are considering the problem

$$Ly = f(x)$$

where L is not Sturm-Liouville. We could solve following the approach in Eq'n (21); alternatively we could convert to Sturm-Liouville first, and then proceed using the nice properties of a self-adjoint operator. So, is the problem self-adjoint or isn't it?? The key observation is that we are no longer solving the same problem. We have transformed to a new operator

$$\hat{L}y = -\frac{d}{dx}(py') + qy$$

which does not satisfy the same equation as the original, that is $Ly = f$ while $\hat{L}y = \mu f$. They are both valid, and must ultimately lead to the same answer in the end.

2.7.3 Further properties

Orthogonality.
Due to the presence of the weighting function, the *orthogonality* relation is

$$\int_a^b y_k(x)y_j(x)r(x)\mathrm{d}x = 0. \tag{28}$$

Eigenvalues.

The functions p, q, r are real, so $\overline{L} = L$. Thus, taking the conjugate of both sides of $Ly_k = \lambda_k r y_k$ gives

$$L\,\overline{y_k} = \overline{\lambda_k}\, r\, \overline{y_k}$$
$$\Rightarrow \langle y_k, L\,\overline{y_k}\rangle = \overline{\lambda_k}\langle y_k, r\overline{y_k}\rangle$$
$$\text{but } \langle y_k, L\,\overline{y_k}\rangle = \langle Ly_k,\ \overline{y_k}\rangle = \lambda_k\langle ry_k,\ \overline{y_k}\rangle = \lambda_k\langle y_k, r\overline{y_k}\rangle \qquad (29)$$

$$\Rightarrow \overline{\lambda_k} = \lambda_k$$

Thus, all eigenvalues are real.

Moreover, if $a \le x \le b$ is a finite domain, then λ's are <u>discrete and countable</u>:
$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k < \cdots$$
with $\lim_{k\to\infty} \lambda_k = \infty$.

Eigenfunctions.

The $\{y_k\}$ are a *complete set*, that is all $h(x)$ with $\int h^2 r\, dx < \infty$ can be expanded as
$$h(x) = \sum c_k y_k(x).$$

Take an inner product with $r(x)y_j(x)$:

$$\langle ry_j, h\rangle = \langle ry_j, \sum c_k y_k\rangle = \sum c_k \langle ry_j, y_k\rangle = c_j \langle ry_j, y_j\rangle$$

$$\Rightarrow c_j = \frac{\int_a^b h(x)y_j(x)r(x)\ dx}{\int_a^b y_j^2(x)r(x)\ dx}$$

Note: I've used $h(x)$ to make clear that we're not talking about the solution to the BVP, rather we are expanding any function that is suitably bounded on the same domain.

2.7.4 Other tidbits

Regular Sturm-Liouville Problems. If the system satisfies all of the above and the additional conditions

- $p(x) > 0$ and $r(x) > 0$ on $a \le x \le b$.
- $q(x) \ge 0$ on $a \le x \le b$.
- BCs have $\alpha_1\alpha_2 \le 0$ and $\alpha_3\alpha_4 \ge 0$,

then all $\boxed{\lambda_k \geq 0}$

Proof: Using $\langle y_k, Ly_k - \lambda_k r y_k \rangle = 0$,

$$-\int_a^b y(py')'\,dx + \int_a^b yqy\,dx - \int_a^b y\lambda ry\,dx = 0$$

$$-\int_a^b y(py')'\,dx + \int_a^b qy^2\,dx - \lambda\int_a^b ry^2\,dx = 0$$

$$-pyy'\Big|_a^b + \int_a^b p(y')^2\,dx + \int_a^b qy^2\,dx - \lambda\int_a^b ry^2\,dx = 0$$

$$\lambda = \left[\int_a^b p(y')^2\,dx + \int_a^b qy^2\,dx - pyy'\Big|_a^b\right] \bigg/ \int_a^b ry^2\,dx \geq 0$$

As a side note, the **Rayleigh quotient**, $R[y] = \langle y, Ly \rangle / \langle y, ry \rangle$, is used extensively in analysis.

3 Green's function

In this section we will devise an alternative approach to viewing and solving linear BVP, using the so-called Green's function.

3.1 Form of the eigenfunction expansion solution

Consider the form of the final solution obtained through the eigenfunction expansion approach. Taking (21) one step further, we have

$$y(x) = \sum_{k=1}^{\infty} \frac{\langle f, w_k \rangle}{\lambda_k \langle y_k, w_k \rangle} y_k(x)$$

(Of course, this requires all $\lambda_k \neq 0$ – we'll treat the case of a zero eigenvalue in Section 4). Let $n_k = \langle y_k, w_k \rangle$ (normalisation), then:

$$
\begin{aligned}
y(x) &= \sum_{k=1}^{\infty} \frac{1}{\lambda_k n_k} \left(\int_a^b f(t) w_k(t)\mathrm{d}t \right) y_k(x) \\
&= \int_a^b \left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k n_k} w_k(t) y_k(x) \right) f(t)\mathrm{d}t \\
&= \int_a^b g(x,t) f(t)\mathrm{d}t
\end{aligned}
$$

where

$$g(x,t) = \sum_{k=1}^{\infty} \frac{w_k(t)y_k(x)}{\lambda_k n_k}. \tag{30}$$

Thus, we have constructed a solution to $Ly = f$ in the form

$$y(x) = \int_a^b g(x,t)f(t) \, dt. \tag{31}$$

The function $g(x,t)$ is called the Green's function (GF), and the form (30) is an eigenfunction expansion of $g(x,t)$.

Of course, if we knew the Green's function, we would have the solution without any need for the expansion, i.e. no need for the eigenfunctions. The goal in this section is to understand the properties of the GF and how to construct it.

Side note: Observe that if $L = L^*$, then $w_k = y_k$ and:

$$g(x,t) = \sum \frac{1}{\lambda_k n_k} y_k(t)y_k(x)$$

In this case $g(x,t) = g(t,x)$, and we have the important connection between a self-adjoint operator and a symmetric Green's function.

3.2 Inverse of differential operator

A nice way to think of the Green's function is in terms of inverting the differential operator. Think about the familiar equation $\mathbf{A}\vec{x} = \vec{b}$ from linear algebra, to be solved for the unknown vector $\vec{x}$. The solution is given by

$$\vec{x} = \mathbf{A}^{-1}\vec{b},$$

i.e. we find the solution by multiplying the inverse of the linear operator (matrix) by the inhomogeneous term. Once you know the inverse operator, you can solve the problem for any given vector $\vec{b}$. In the context of BVP's, L is a differential operator, so it stands to reason that the inverse operator involve integration, hence the form (31). Constructing the Green's function is analogous to finding the inverse of the matrix, once we have g we can write down the solution (31) for any forcing function $f(x)$.

3.2.1 An example

There are numerous ways to construct a Green's function. We've just seen one: the eigenfunction expansion. In fact, we've already seen two! The method of variation of parameters gives the Green's function in a piecewise form.

Let's take a simple example and look at the behaviour. Consider the BVP:

$$Ly \equiv -y'' = f(x), 0 < x < 1$$
$$y(0) = y(1) = 0$$

(32)

The GF via variation of parameters is given by

$$g(x, \xi) = \begin{cases} (1 - \xi)x & 0 < x < \xi \\ (1 - x)\xi & \xi < x < 1. \end{cases}$$

(33)

The following properties are easily checked:

- The GF satisfies $Lg = 0$ if $x \neq \xi$[5]

- $g(x, \xi)$ satisfies the boundary conditions *as a function of x*.

- g is continuous on the whole interval $[0, 1]$

- g is differentiable everywhere *except at $x = \xi$*, where it suffers a jump in the derivative.

These properties are in fact always true of the GF of a second order linear operator.[6] To make sense of this, and to build some physical intuition, we shall need the notion of the delta function.

3.3 Green's function via delta function

To fix the context, consider stationary heat conduction in a rod:

$$-y''(x) = f(x) \qquad 0 < x < 1 \tag{34}$$
$$y(0) = 0, \;\; y(1) = 0. \tag{35}$$

where $y(x)$ is the temperature field and $f(x)$ is a given heat source density.

---

[5]Here by L we mean the operator acting on the x variable, i.e. derivatives are with respect to x – this is sometimes written $L_x g(x, \xi)$ to clarify.

[6]Note, however, that the function $y(x)$ satisfying $Ly = f$ *is* continuously differentiable assuming continuously differentiable f, meaning that the integration with $f(x)$ smooths out the discontinuity in g.

3.3.1 Delta function

The function $f(x)$ describes any heat added or removed from the system by the outside world. As a simple scenario, consider a *point heat source*, say located at the middle of the rod. Physically, this would correspond to applying heat at a single point only. How would we describe such a situation mathematically? What should we use for the function $f(x)$?

The notion of a point source is described by the "delta function" δ, characterised by properties

$$\delta(x) = 0 \quad \forall x \neq 0, \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{36}$$

The first property captures the notion of a point function. The second property constrains the area under the curve (which you might think of as infinitely thin and infinitely high). This is an idealized point source at $x = 0$, a point source at $x = a$ would be given by $\delta(x - a)$.

The problem is that no classical function satisfies (36) (think: any function that is non-zero only at a point is either not integrable or integrates to zero).

3.3.2 Approximating the delta function

One way around this is to replace δ by an approximating sequence of increasingly narrower functions with normalized area, i.e. $f_n(x)$ where

$$\int_{-\infty}^{\infty} f_n(x)\mathrm{d}x = 1 \quad \forall n, \quad \lim_{n \to \infty} f_n(x) = 0 \quad \forall x \neq 0.$$

Example: "hat" functions

$$f_n(x) = \begin{cases} 0 & \text{for } |x| > 1/n \\ n/2 & \text{for } |x| \leq 1/n \end{cases} \tag{37}$$

You can verify the $f_n(x)$ approach $\delta(x)$ as $n \to \infty$.

3.3.3 Properties of delta function

We have defined δ by (36). We can use the approximating functions to obtain further properties.

Sifting property. What happens when δ is integrated against another function?

Let $f(x)$ be a continuous function, and $F(x) = \int^x f(s)ds$ its antiderivative. Now consider approximating sequences:

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)\mathrm{d}x = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x-a)f(x)\mathrm{d}x,$$

and if f_n are the hat functions (37),

$$= \lim_{n\to\infty} \int_{a-1/n}^{a+1/n} \frac{n}{2} f(x)\mathrm{d}x = \lim_{n\to\infty} \frac{F(a+(1/n)) - F(a-(1/n))}{2/n}$$

$$= \lim_{s\to 0} \frac{F(a+s) - F(a-s)}{2s} = F'(a) = f(a).$$

Thus, we have

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)\mathrm{d}x = f(a) \quad \text{if } f \text{ is continuous at } a. \tag{38}$$

In particular,

$$\int_{-\infty}^{\infty} \delta(x)f(x)\mathrm{d}x = f(0) \quad \text{if } f \text{ is continuous at } x = 0. \tag{39}$$

Thus, the delta function can be seen to sift out the value of a function at a particular point.

Antiderivative of $\delta(x)$. The antiderivative of the delta function is the so-called Heaviside function,

$$\int_{-\infty}^{x} \delta(s)\mathrm{d}s = H(x) \equiv \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases} \tag{40}$$

Note that (40) follows by integrating the sequence of approximating functions and showing that the limit is the Heaviside function. That is, if $H_n(x) = \int_{-\infty}^{x} f_n(s)ds$, then $\lim_{n\to\infty} H_n(x) = H(x)$. (We leave this detail as an exercise!)

3.3.4 Point heat source

Let's return to the heat conduction BVP with a point heat source of unit strength at the centre of the rod:

$$-y''(x) = \delta(x - 1/2), \quad 0 < x < 1 \tag{41}$$

$$y(0) = y(1) = 0. \tag{42}$$

Since $\delta(x - 1/2) = 0 \;\; \forall x \neq 1/2$, this implies

$$-y''(x) = 0, \quad 0 < x < 1/2, \; 1/2 < x < 1. \tag{43}$$

We can easily solve (43) in each of the two separate domains $[0, 1/2)$ and $(1/2, 1]$ and then apply the BC (42). But be careful: there are two constants of integration for each domain, meaning *four* unknown constants total, and only *two* boundary conditions.

As you might expect (since $\delta(x - 1/2)$ has vanished from (43)), the extra two conditions come in at $x = 1/2$. To derive the extra conditions, imagine integrating equation (41) across $x = 1/2$:

$$\int_{1/2-}^{1/2+} -y''(x) \; dx = \int_{1/2-}^{1/2+} \delta(x - 1/2) \; dx, \tag{44}$$

where $1/2-$ $(1/2+)$ signifies just to the left (right) of $1/2$. Using property (36) of the delta function, we have

$$-y']_{1/2-}^{1/2+} = 1 \quad \Rightarrow \quad y'(1/2+) - y'(1/2-) = -1. \tag{45}$$

That is, the presence of the delta function defines a *jump condition* on y'. [7] The other extra condition needed comes as a requirement that $y(x)$ is continuous across the point source, that is

$$y]_{1/2-}^{1/2+} = 0. \tag{46}$$

More on this condition below. Solving Equations (43), (42) along with extra conditions (45) and (46), we obtain the solution

$$u(x) = \begin{cases} \frac{x}{2} & 0 < x < 1/2 \\ -\frac{x}{2} + \frac{1}{2} & 1/2 < x < 1. \end{cases} \tag{47}$$

3.3.5 Green's function construction

To motivate the construction of the Green's function, consider the heat conduction problem with an arbitrary heat source:

$$-y''(x) = f(x), \quad 0 < x < 1 \tag{48}$$
$$y(0) = y(1) = 0. \tag{49}$$

---

[7]Here, $y(\xi-) = \lim_{x \uparrow \xi} y(x)$, and $y(\xi+) = \lim_{x \downarrow \xi} y(x)$

Imagine now describing f by a distribution of point heat sources with varying strength; that is at point $x = \xi$ we imagine placing the point source $f(\xi)\delta(x - \xi)$.

The idea of the Green's function is to introduce such an extra parameter ξ, and consider the system

$$-g''(x, \xi) = \delta(x - \xi), \quad 0 < x < 1 \tag{50}$$
$$g(0, \xi) = g(1, \xi) = 0. \tag{51}$$

Note that prime denotes differentiation with respect to x, while ξ is more like a place-holding variable. So, we have replaced $f(x)$ by a delta function, in order to solve for the Green's function $g(x, \xi)$.

We have seen how to solve (50), (51) in the last section. The Green's function is

$$g(x, \xi) = \left\{ \begin{array}{ll} (1 - \xi)x & 0 < x < \xi \\ (1 - x)\xi & \xi < x < 1. \end{array} \right. \tag{52}$$

You can notice that this is exactly the solution (50) one would obtain via variation of parameters.

How to get back to the solution of (48), (49)? For each ξ, the Green's function gives the solution if a point heat source of unit strength were placed at $x = \xi$. Conceptually, then, to get the full solution we must "add up" the point sources, scaled by the value of the heat source at each point:

$$y(x) = \int_0^1 g(x, \xi) f(\xi) \, d\xi. \tag{53}$$

To verify that this is indeed a solution, we can plug (53) into (48):

$$-y''(x) = \int_0^1 -g''(x, \xi) f(\xi) \, dx = \int_0^1 \delta(x - \xi) f(\xi) \, dx = f(x) \; \checkmark \tag{54}$$

3.4   General linear BVP

We now consider a general nth order linear BVP with arbitrary continuous forcing function,

$$Ly(x) = a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = f(x) \tag{55}$$

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for $a < x < b$, where each $a_i = a_i(x)$ is a continuous function, and moreover $a_n(x) \neq 0 \ \forall x$[8]. Along with (55) are n boundary conditions, each a linear combination of y and derivatives up to $y^{(n-1)}$, evaluated at $x = a, b$. For instance, in the case $n = 2$, the general form is:

$$\begin{aligned} B_1 y &\equiv \alpha_{11} y(a) + \alpha_{12} y'(a) + \beta_{11} y(b) + \beta_{12} y'(b) = \gamma_1 \\ B_2 y &\equiv \alpha_{21} y(a) + \alpha_{22} y'(a) + \beta_{21} y(b) + \beta_{22} y'(b) = \gamma_2. \end{aligned} \tag{56}$$

3.5 General Green's Function

In the same way as in Section 3.3.4, to solve (55) with homogeneous BC

$$B_i y = 0, \quad i = 1 \ldots n - 1,$$

we first determine the Green's function by solving

$$\begin{aligned} Lg(x, \xi) &= \delta(x - \xi), \quad a < x < b \\ B_i g &= 0. \end{aligned} \tag{57}$$

As before,

$$Lg(x, \xi) = \delta(x - \xi)$$

implies

$$Lg(x, \xi) = 0 \quad \text{on } a < x < \xi, \ \xi < x < b,$$

i.e. we have a homogeneous problem to solve on two separate domains. As before, we require extra conditions, which come by integrating $Lg(x, \xi) = \delta(x - \xi)$ across $x = \xi$:

$$\int_{\xi-}^{\xi+} a_n g^{(n)}(x, \xi) + \cdots + a_0 g(x, \xi) \, d\xi = \int_{\xi-}^{\xi+} \delta(x - \xi) \, d\xi. \tag{58}$$

The right hand side clearly integrates to one. If we were to perform an integration by parts on the first term of the left hand side, we would obtain

$$a_n(x) g^{(n-1)}(x, \xi)]_{\xi-}^{\xi+} + \int_{\xi-}^{\xi+} (a_{n-1} - a_n') g^{(n-1)} + \cdots + a_0 g(x, \xi) \, d\xi = 1.$$

This equation is balanced by setting a jump condition on the $n - 1$st derivative:

$$g^{(n-1)}(x, \xi)]_{\xi-}^{\xi+} = 1/a_n(\xi),$$

---

[8]We'll return to the case where $a_n(x) = 0$ somewhere in the domain later in the course.

and taking all lower derivatives to be continuous across $x = \xi$:

$$g^{(j)}(x,\xi)]_{\xi-}^{\xi+} = 0, \quad j = 0, 1, \ldots n - 2.$$

Once the Green's function is determined, the solution to the BVP is given by

$$y(x) = \int_a^b g(x,\xi)f(\xi)\,d\xi. \tag{59}$$

3.6 Another view

There is one more way of viewing the GF. Start from $Ly(x) = f(x)$, and take an inner product with $G(x,\xi)$ on both sides of the equation[9]. We are not assuming we know G, rather we want to find properties it should satisfy for us to solve the equation. We obtain

$$\langle Ly, G \rangle = \langle G(x,\xi), f(x) \rangle = \int_a^b G(x,\xi)f(x)\,dx. \tag{60}$$

(Note the integration is over x). Now, using the adjoint, we can write

$$\langle Ly, G \rangle = \langle y, L^*G \rangle \tag{61}$$

The idea now is to isolate y. This can be accomplished if

$$L^*G(x,\xi) = \delta(x - \xi) \tag{62}$$

in which case the left hand side leaves just $y(\xi)$, and we have the solution

$$y(\xi) = \int_a^b G(x,\xi)f(x)\,dx. \tag{63}$$

Comparing with our previous construction, here the big difference is that the GF is constructed through the adjoint operator – hence we will refer to this as the adjoint Green's function. Compare the form of solution with the form (31):

$$y(x) = \int_a^b g(x,t)f(t)\,dt, \tag{64}$$

we see the subtle difference that in (63) we integrate over the first variable of the adjoint GF, and the second variable of the GF. For a self-adjoint operator, the constructions are the same and we must get the same GF, and indeed as we've stated, the GF for a self-adjoint operator is symmetric.

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[9]G will be the Green's function, but not quite the same one we've constructed, so I am differentiating by using capital G.

4 Fredholm Alternative

We have now seen two different ways to solve linear BVPs. So far, we have been happily going along, assuming that the solutions we construct are *the, one and only*, solutions. But are we sure? Now it is time to address the important question of existence and uniqueness.

Existence: Does a solution actually exist? Will the method we employ "work," or will it for instance lead to contradictions?

Uniqueness: We've found a solution, it exists, great. Is it the only one?

From a mathematical point of view, these are incredibly important questions. They can be even more important from a physics/applied mathematics standpoint. If you're modelling a physical situation with a particular differential equation, you'd like to think that you can find a single answer that has physical meaning and that would agree with experiment.
To think about: What would non-existence mean in a physical context? What about non-uniqueness?

These are important questions, and whole branches of mathematics have been developed to think about them. Here, we'll just look at one very useful theorem in this arena: the Fredholm Alternative Theorem (FAT).

4.1 A closer look at SL solution

Let's look more closely at the solution we obtained in Section 2.7.1 for a SL-BVP. The last step in obtaining the coefficients c_k of

$$y = \sum_k c_k y_k$$

was

$$\lambda_k c_k \langle y_k, r y_k \rangle = \langle f, y_k \rangle. \tag{65}$$

But what happens if there is an eigenvalue equal to zero, say $\lambda_0 = 0$? We potentially have a problem, as this equation would read

$$0 \times c_0 = \langle f, y_0 \rangle.$$

If so, there are two possibilities:

1. $\langle f, y_0 \rangle \neq 0$. In this case, we have a contradiction, and we are forced to conclude that no solution exists.

2. $\langle f, y_0 \rangle$ does actually equal zero. In this case, no contradiction, but we don't get any way to solve for c_0 either. Hence

$$y = c_0 y_0(x) + \sum_{k=1}^{\infty} c_k y_k(x)$$

   where c_k are calculated values for $k = 1, 2, \ldots$, but c_0 is arbitrary, leading to an *infinite set of solutions.*

So, if there is a zero eigenvalue (meaning $\lambda_0 = 0$ yields a non-trivial solution), it seems that either we get non-existence, or we get existence but non-uniqueness.

And if there isn't a zero eigenvalue? If there is no eigenfunction corresponding to $\lambda_0 = 0$, then (65) would never yield any problems, we could always divide by λ_k to get well defined c_k, and the expansion would have no issues. Thus, we would have both existence and uniqueness.

4.2   Zero eigenvalue

The question of the zero eigenvalue is a special case, as we are really asking whether the *homogeneous* system

$$Ly = 0$$

has a non–trivial solution $y_0(x)$.

4.3   FAT

What we've just seen above for a SL-BVP in fact holds much more widely. For general (not necessarily self-adjoint) ODE-BVPs, the statement of FAT reads:

Exactly one of the two alternatives holds:

I. **EITHER** the homogeneous adjoint problem

$$L^* w_0 = 0, \ \ BC_1^* = 0, \ BC_2^* = 0$$

has a non-trivial solution

II. **OR** the inhomogeneous problem

$$Ly = f, \ \ BC_1 = \alpha_1, \ BC_2 = \alpha_2$$

has a unique solution for any f, α_1, α_2

Notes:

1. Note the exclusive "or". Exactly one of the alternatives is true.

2. You'll see I've put in inhomogeneous boundary conditions. If we are in Case II, with no homogeneous solution, then we can construct a solution with inhomogeneous BC either by decomposing or by putting BC directly into the eigenfunction construction.

In Case I, there are two subcases. Suppose we have inhomogeneous BCs, and we solve the problem by putting the BC's directly into the eigenfunction expansion. Since there is an eigenvalue $\lambda_0 = 0$, we will have an equation that looks like:

$$0 \times c_0 = \langle f, w_0 \rangle + \text{'stuff'}_0$$

where 'stuff'$_0$ has come from the inhomogeneous boundary conditions. The two possibilities are:

1. If the RHS is non-zero, then we have a contradiction, and thus no solution exists.

2. If the RHS=0, there is no contradiction, but we have no information on c_0 either, thus the solution is valid for any constant c_0, and we have existence but non-uniqueness.

4.4 Homogeneous vs inhomogeneous BC

With homogeneous boundary conditions, FAT has a very 'nice' form, easily stated in words. It says that to have a unique solution, the adjoint homogeneous problem must have only the trivial solution; otherwise, if there is a non-trivial solution w_0, then the *solvability condition* to have a solution (but non-unique!) is that the forcing function $f(x)$ must be orthogonal to the homogeneous solution w_0.

With inhomogeneous BC, the criteria for a unique solution is the same, but when the non-trivial w_0 exists, the solvability condition is not so cute, due to the 'stuff'$_0$ arising from the BCs. Note that if we had tried to decompose the solution, i.e. separate the boundary conditions into a problem $Ly = 0$, $BC \neq 0$, then in looking at $Ly = f$, $BC = 0$, we would arrive at the wrong solvability condition, since there would be no mention of boundary conditions! In other words, decomposing the solution could lead to incorrect conclusions! Which leads to the following...

Health warning: If you have a problem with *inhomogeneous boundary conditions* AND there is a *zero eigenvalue*, do not try to decompose the solution. Incorporate the boundary conditions directly into the eigenfunction expansion!

4.5 Examples

Ex. 1
Solve $y'' + y = f$, $y(0) = 0$, $y(\pi) = 0$. This is self-adjoint and has the zero eigensolution $y_0 = \sin x$. Then

$$
\begin{aligned}
\langle y'', \sin x \rangle &= \int_0^\pi y'' \sin x \, dx \\
&= -\int_0^\pi y \sin x \, dx \quad \text{(by parts, twice)} \\
&= -\langle y, \sin x \rangle
\end{aligned}
$$

There is a solution only if

$$\langle f, \sin x \rangle = 0,$$

in which case $y + c \sin x$ is a solution for all c.

Ex. 2a
Solve $y'' = f(x)$ with $0 < x < 1$, $y(0) = 0$ and $y'(1) = 7$. The zero eigenvalue

adjoint problem is $L^*w_0 = w_0''(x) = 0$ (this problem is also self-adjoint), with $w_0(0) = w_0'(1) = 0$. This only has the trivial solution $w_0 \equiv 0$, so the full problem has a unique solution for any $f(x)$.

Ex. 2b

Same problem, but change the BC to $y'(0) = 0$ and $y'(1) = \beta$, and let $f(x) = 3$. This time we get $w_0(x) = 1$. Thus:

$$\langle y'', w_0 \rangle = \langle f, w_0 \rangle$$
$$\Rightarrow \int_0^1 y'' \mathrm{d}x = 3$$
$$\Rightarrow y'|_0^1 = 3$$

The BC's give that $y'(1) - y'(0) = \beta$, and thus if $\beta \neq 3$, we have a contradiction and no solution exists, while if $\beta = 3$, we have a non-unique solution.

4.6   FAT - Linear algebra version

The Fredholm alternative can also be expressed as a theorem of linear algebra. It addresses the question: when does $Ax = b$ have a unique solution?, where

$$\left. \begin{array}{l} A \in \mathbb{R}^{n,n} : n \times n \text{ matrix} \\ b \in \mathbb{R}^n : n \text{ dim, column vector} \end{array} \right\} \text{ given}$$

$$x \in \mathbb{R}^n : n \text{ dim column vector (unknown)}$$

FAT says:

Exactly one of the two alternatives holds:

I. Either: $A^T y = 0$ has a non-trivial solution $y \neq 0$.

II. Or: $Ax = b$ has a unique solution (has a solution, and it is unique).

If $A^T y = 0$ has solutions $y \neq 0$, then $Ax = b$ has either no or multiple solutions.

Distinguish cases by solvability condition.

Let $y_1, y_2, \ldots, y_N$ be a basis of $A^T y = 0$.

(a) If $y_k^T b = 0$ for <u>all</u> k, then $Ax = b$ is solvable, and the solution space has dimension N.

(b) If $y_k^T b \neq 0$ for one or more k, then $Ax = b$ has no solutions.

Example

(1)

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Look at $A^T y = 0$ (GE=Gaussian Elimination)

$$\left(\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 4 & 0 \end{array}\right) \xrightarrow{\text{GE}} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -2 & 0 \end{array}\right) \Rightarrow \begin{array}{l} y_1 + 3y_3 = 0 \\ -2y_2 = 0 \end{array}$$

$\Rightarrow y = 0$ i.e. $A^T y = 0$ only has a trivial solution. Therefore, FAT I is false, thus FAT II is true i.e. $Ax = b$ has a unique solution (for any b).

(2)

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A^T y = 0 : \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right) \xrightarrow{\text{GE}} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow y_1 + 3y_2 = 0$$

Thus, a non-trivial solution for $A^T y = 0$ is: $y_0 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

Therefore I is true, thus II of FAT is false:
$Ax = b$ has either none or multiple solutions. Which of these two cases applies? That depends on b. Since y_0 forms a basis for the one dimensional null space $A^T y = 0$, the solvability condition is $y_0^T b = 0$.

Alternatively, we could see this by GE for $Ax = b$.

$$\left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 6 & b_2 \end{array}\right) \xrightarrow{\text{GE}} \left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 3b_1 \end{array}\right) \Rightarrow \begin{array}{l} x_1 + 2x_2 = b_1 \\ 0 = b_2 - 3b_1 \end{array}$$

So, no solution if $b_2 - 3b_1 \neq 0$, and a one-dimensional space of solutions if $b_2 - 3b_1 = 0$.
Note:

$$b_2 - 3b_1 = (-3, 1) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = y^T b$$

4.7 FAT and Variation of parameters

To see how Fredholm Alternative ties in with the Variation of parameters recipe, suppose y_1 satisfies $Ly_1 = 0$ and both boundary conditions (i.e. there exists a zero eigenvalue), say $y_1(a) = y_1(b) = 0$. We let y_2 be a second, linearly independent solution (which means that y_2 can't satisfy either boundary condition, else we'd have the Wronskian $W = 0$). Then consider

$$y = c_1 y_1 + c_2 y_2.$$

Applying $y(a) = 0$ requires $c_2(a) = 0$, so we have

$$c_2(x) = \int_a^x \frac{f(t)y_1(t)}{P_2(t)W(t)} \, dt,$$

as before (Eqn (13)). But then $y(b) = 0$ requires $c_2(b) = 0$, i.e.

$$\int_a^b \frac{f(t)y_1(t)}{P_2(t)W(t)} \, dt = 0$$

this is the solvability condition. If this solvability condition holds (FAT I), then we can construct our solution, but we have no information on the additive constant in c_1, i.e. existence but not uniqueness.

As an **example**, consider the problem

$$y'' + y = f(x), \ y(0) = 0 = y(\pi), \tag{66}$$

so H is $y'' + y = 0$.

We try to follow the method: choose y_1 so that $y_1(0) = 0$, e.g. $y_1 = \sin x$; but then $y_1(\pi) = 0$ too. So y_1 is the zero eigenfunction. We revert to Variation of Parameters, choosing a second, linearly independent, solution y_2 of H, say $y_2 = \cos x$. Then $W = -1$, $P_2 = 1$ and $y = c_1(x)y_1(x) + c_2(x)y_2(x)$ with c_i as in (9). Now we try to impose the boundary conditions:

$$y(0) = 0 \Rightarrow c_2(0) = 0 \Rightarrow c_2 = -\int_0^x f(t) \sin t \, dt$$

by (9); but now

$$y(\pi) = -c_2(\pi)$$

so for $y(\pi) = 0$, we must have

$$\int_0^\pi f(t) \sin t \, dt = 0. \tag{67}$$

4.8 FAT and Green's functions

As we've seen, Variation of Parameters is just one way to construct the Green's Function. The Green's function approach gave the solution to $Ly = f$ as

$$y = \int_a^b g(x, \xi) f(\xi) \, d\xi.$$

So, if the GF approach works, i.e. if we can find g, then we have both existence and uniqueness. Clearly, if there is a zero eigenvalue, something must go wrong with any construction of the GF.

Consider the delta function formulation for the GF:

$$Lg(x, \xi) = \delta(x - \xi),$$

and let's apply FAT, thinking of ξ as a dummy variable. If there is a zero eigenvalue, then there exists non-trivial $w_0(x)$ for which $L^* w_0 = 0$. Then we are in case I and $Lg = \delta$ does not have a unique solution, and the solvability condition for any solution to exist is

$$\langle w_0(x), \delta(x - \xi) \rangle = 0$$

which clearly does not hold since

$$\langle w_0(x), \delta(x - \xi) \rangle = w_0(\xi)$$

and $w_0 \not\equiv 0$. Thus, we can't construct the GF. If the solvability condition is satisfied, we can construct a so-called *Modified Green's function* can be constructed, yielding a non-unique solution, but we won't go into details here.

5 Singular points of differential equations

In this section we will seek solutions of the nth order linear differential equation

$$Ly = y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y^{(n)}(x) + p_0(x)y(x) = 0, \quad (68)$$

in the form of a *series expansion* in the neighbourhood of $x = x_0$. How we proceed, and the nature of the solution, depends on how well-behaved the functions $p_j(x)$ are around x_0.

5.1 Ordinary points

The point x_0 is an ordinary point of the ODE if all $p_j(x)$ are analytic there, i.e. they can be expanded as a convergent power series. The procedure in this case is pretty straightforward: (i) write $y(x) = \sum_0^\infty a_k x^k$ as a power series, (ii) plug into (68), using the power series expansions of each of the p_j, then (iii) obtain a sequence of equations for the coefficients a_k that can be solved recursively.

This is the simplest (and least interesting) case, so we won't really spend any time on it, but it is worth noting a few things about ordinary points:

- All n linearly independent solutions of (68) are analytic at x_0.

- The radius of convergence of the series solution $\geq$ distance (in $\mathbb{C}$) to next singular point.

Example:
$$(x^2 + 1)y' + 2xy = 0$$

Here $x_0 = 0$ is an ordinary point. Nearest singular points are $x = \pm i$, distance 1 from $0 \Rightarrow$ radius of convergence ≥ 1.

Here we can obtain the solution $\frac{1}{(1+x^2)}$ via easier routes, but we note that

$$\frac{1}{(1+x^2)} = 1 - x^2 + x^4 - \dots$$

with a radius of convergence $= 1$, is the series solution one would obtain.

5.2 Singular points

The point x_0 is called **a singular point of the ODE** if one of the $p_j(x)$ is not analytic there.

In this case, the general solution y <u>may</u> have a singularity at $x = x_0$ <u>(but not necessarily)</u>. This means that the general solution y may not be analytic at x_0: y or its derivatives can "blow-up" as $x \to x_0$.

Where do ODEs with singular points arise?

- As we will see later, equations with singular points at $x = 0$ commonly arise from linear PDEs in polar/spherical co-ordinates (where $x = 0$ corresponds to radius $r = 0$).

- SL-BVPs with a singular point at the boundary - such problems often require the solution to be bounded at the boundary point (rather than prescribing a specific value).

Note: $x_0 = \infty$ can also be classified as an ordinary or singular point by changing the independent variable via the substitution $t = 1/x$, $u(t) = y(x)$, (i.e. $\mathrm{d}y/\mathrm{d}x = -t^2 \mathrm{d}u/\mathrm{d}t$, etc.) and classifying the point $t = 0$ for the resulting ODE for $u(t)$.

Some Examples

(a) $y'' = e^x y$: every x_0 is an ordinary point

(b) $x^5 y''' = y$: $x_0 = 0$ is a singular point, every $x_0 \neq 0$ is an ordinary point.

Let's consider more closely a simple example to see the types of solutions near singular points. Consider the equation:

$$y' + x^{-m} y = 0, \quad m \geq 0 \text{ an integer}$$

The general solution can be found via separation of variables, and depends on the value of m:

1. For $m = 0$, the point $x = 0$ is ordinary and $y(x) = C \exp(-x)$. This function can be expanded into a power series at $x = 0$ which converges for all $x \in \mathbb{C}$.

2. For $m = 1$, $y(x) = C/x$. This solution clearly has a singularity at $x = 0$, but a rather benign one (a simple pole).

3. For $m = 2$ (and similarly for larger m), $y(x) = C \exp(1/x)$; this solution has a very strong singularity: $x = 0$ is an essential singularity of $\exp(1/x)$ in the complex plane.

This example suggests that the solution at a singular point of an ODE tends to have a stronger singularity the higher the order of the poles in the coefficients in front of the lower order terms of the ODE. In fact, this is the key idea behind the classification of singular points:

Regular singular points: x_0 is a regular singular point, if all $\tilde{p}_j(x) \equiv p_j(x)(x - x_0)^{n-j}$ are analytic at $x = x_0$ (for $j = 0, \ldots, n-1$).

Irregular singular points: all other singular points.

For regular singular points, you might think of it as the singularities are not "too bad", we are essentially able to remove the trouble in p_j by multiplying by a power of $(x - x_0)$ decreasing with the degree of derivative. In this case, a modification of the power series approach can be used.

For irregular singular points, though, there is no general theory!

Cauchy-Euler. One of the simplest and most instructive examples is the Cauchy-Euler equation

$$x^2 y'' + axy + by = 0,$$

which clearly has a regular singular point at $x = 0$. As you've seen before, the general solution can be found via the ansatz $y = x^m$. The characteristic equation for m is $m(m - 1) + am + b = 0$. There are two cases:

(i) The characteristic equation has two distinct roots m_1 and m_2. Then, the general solution is:

$$y(x) = C_1 x^{m_1} + C_2 x^{m_2}.$$

(ii) The characteristic equation has a double root m. Then, the general solution is:

$$y(x) = C_1 x^m + C_2 x^m \ln(x).$$

Note that if the roots are two distinct non-negative integers, then the general solution in (i) is analytic (even though the ODE has a singular point). In general, however, the behaviour as $x \to 0$ is a fractional or even complex power of x.

This behaviour carries over to the general situation for regular singular points, except that the functions x^m are multiplied by an analytic function (i.e. a power series in x). Next, we'll look at the general theory for regular singular points.

5.3 Frobenius method for 2nd order ODEs

From now on, we'll restrict to 2nd order equations. If the ODE

$$Ly \equiv y'' + P(x)y' + Q(x)y = 0 \tag{69}$$

has a regular singular point at $x = x_0$, then

$$p(x) := P(x)(x - x_0)$$

and

$$q(x) := Q(x)(x - x_0)^2$$

are analytic, i.e.

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n \tag{70}$$

$$q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n \tag{71}$$

The idea is to seek a solution in the form of a *Frobenius series*

$$y(x) = (x - x_0)^\alpha \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{72}$$

(In terms of the Cauchy-Euler example, α is playing the role of m and $\sum a_n (x - x_0)^n$ is the analytic function with coefficients a_n to be determined.) We may assume that $a_0 = 1$ (by choosing α appropriately and normalizing). Now plug in and equate coefficients. At the lowest power $((x - x_0)^{\alpha-2}$, we find

$$\alpha(\alpha - 1) + p_0\alpha + q_0 = 0$$

This polynomial plays an important role, we will denote it

$$F(\alpha) = \alpha(\alpha - 1) + p_0\alpha + q_0.$$

The equation $F(\alpha) = 0$ is called the **indicial equation**, it determines the possible indicial exponents α_1, α_2. Note that these exponents can be complex! We'll order them such that $\Re(\alpha_1) \geq \Re(\alpha_2)$.

Let's carry on with equating coefficients of powers of $(x - x_0)$. We find after some algebra that the coefficients of $(x - x_0)^{n+\alpha-2}$ satisfy

$$F(\alpha + n)a_n = -\sum_{k=0}^{n-1} [(\alpha + k)p_{n-k} + q_{n-k}]a_k \tag{73}$$

Setting $\alpha = \alpha_1$, we know that $F(\alpha_1 + n) \neq 0$ for any integer $n \geq 1$ (do you see why?), thus we can use (73) to solve for all the coefficients a_n, and we obtain one solution

$$y_1(x) = (x - x_0)^{\alpha_1} \underbrace{\sum_{n=0}^{\infty} a_n (x - x_0)^n}_{A(x)}. \qquad (74)$$

Thus the **first solution** can always be expressed as a Frobenius series with indicial exponent $\alpha = \alpha_1$.

For the **second solution**, we have to distinguish between several cases and sub-cases[10].

<u>Case I:</u> $\alpha_1 - \alpha_2$ is not an integer (in particular $\neq 0$).
In this case, $F(\alpha_2 + n) \neq 0$ for all $n \geq 1$. Thus we can solve (73) for all coefficients – let's call them b_n to distinguish from previous coefficients.
Thus, we obtain with no problems a second solution also as a Frobenius series, with indicial exponent α_2,

$$y_2(x) = (x - x_0)^{\alpha_2} \underbrace{\sum_{n=0}^{\infty} b_n (x - x_0)^n}_{B(x)}$$

<u>Case IIa:</u> $\alpha_1 = \alpha_2$
In the case of a double root we clearly only get one solution with the Frobenius method, and we have to multiply by logs to get a second solution (similar to the case of a double root in Cauchy-Euler). In particular, the second solution is of the form

$$y_2(x) = y_1(x) \ln(x - x_0) + (x - x_0)^{\alpha_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

where y_1 is the first solution. We can determine the c_n in the usual manner. A derivation of this form can be done using the so-called derivative method, which is outlined in Section 5.4.

<u>Case IIb:</u> $\alpha_1 - \alpha_2 = N$, where $N > 0$ is an integer. In this case, we will potentially run into trouble in (73) at $n = N$. There are two possibilities:

---

[10]and sub-sub-cases. It's a bit of a headache...

(i) For $n = N$, RHS $\neq 0$ in (73). Then we have a contradiction, and the solution method doesn't work. To get a second solution, we use the same form as Case IIa:

$$y_2(x) = y_1(x) \ln(x - x_0) + (x - x_0)^{\alpha_2} \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

and determine the c_n by substituting into the ODE. Note that the indicial exponent for the second term is α_2 (whereas y_1 is given by the Frobenius series using the exponent α_1).

(ii) For $n = N$, RHS $= 0$ in (73).
There is no contradiction, but any choice for a_n (or b_n) will satisfy (73) $\Rightarrow$ 2nd solution has Frobenius form

$$y_2(x) = (x - x_0)^{\alpha_2} \underbrace{\sum_{n=0}^{\infty} b_n (x - x_0)^n}_{B(x)}. \tag{75}$$

where b_0 can be chosen to be $b_0 = 1$ and b_N is also arbitrary. Notice that changing b_N changes (75) by multiples of y_1.

5.3.1 Example

Find a series solution about the regular singular point $x_0 = 0$ for the differential equation
$$4x^2 y'' + 4xy' + (4x^2 - 1)y = 0. \tag{76}$$

Step 1 Assume a solution of form

$$y = x^\alpha \sum_{k=0}^{\infty} a_k x^k \tag{77}$$

with the assumption $a_0 \neq 0$. Compute the corresponding series for y', y'' by differentiating "term by term".

Step 2 Plug the series into the ODE and multiply everything out.

$$0 = \underbrace{\sum_{k=0}^{\infty} 4(k+\alpha)(k+\alpha-1)a_k x^{k+\alpha}}_{4x^2 y''} + \underbrace{\sum_{k=0}^{\infty} 4(k+\alpha)a_k x^{k+\alpha}}_{4xy'} - \underbrace{\sum_{k=0}^{\infty} a_k x^{k+\alpha}}_{y} + \underbrace{\sum_{k=0}^{\infty} 4a_k x^{\alpha+k+2}}_{4x^2 y}$$
$$\tag{78}$$

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The indicial equation comes from the balance at lowest order, in this case x^α:

$$F(\alpha) = 4\alpha^2 - 1. \tag{79}$$

Step 3 The indicial exponents are the roots of F

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = -\frac{1}{2} \tag{80}$$

Step 4 Shift the terms in the series above in oder to combine all terms into one series, i.e. the goal is to obtain a form

$$\sum [\text{stuff not involving x}] x^{\text{something}} + \text{possibly extra terms from start of series}$$

For this example, we need only shift the index in the last sum, so all series have sum with $x^{\alpha+k}$. Thus, writing

$$\sum_{k=0}^{\infty} 4a_k x^{\alpha+k=2} \overset{n=k+2}{=} \sum_{n=2}^{\infty} 4a_{n-2} x^{n+\alpha}$$

we obtain

$$0 = a_0 F(\alpha) x^\alpha + a_1 (4\alpha^2 + 8\alpha + 3) x^{\alpha+1} + \sum_{n=2}^{\infty} [(4(n+\alpha)^2 - 1)a_n + 4a_{n-2}] x^{n+\alpha} \tag{81}$$

We have chosen the α so that the equation balances at x^α, and hence a_0 is free. Balancing at all other orders will determine the coefficients a_n

Step 5 Treat $\alpha = \alpha_1$ first. Setting $\alpha = \alpha_1 = 1/2$ in (81), we obtain

$$a_1 = 0, \quad a_n = \frac{-1}{n(n-1)} a_{n-2}, \ n = 2, 3, \ldots$$

Step 6 Use the recursion formula to determine a formula for the a_k in terms of a_0. A good idea is to write out a few terms, and look for a pattern.

$$a_2 = \frac{-1}{2 \cdot 3} a_0$$
$$a_3 = 0$$
$$a_4 = \frac{-1}{4 \cdot 5} a_2 = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} a_0 \tag{82}$$
$$\ldots$$
$$a_{2k} = \frac{(-1)^k a_0}{(2k+1)!}, \quad a_{2k+1} = 0, \quad k = 1, 2, 3 \ldots$$

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Step 7 Input the formula for the coefficients to obtain the first solution

$$y_1(x) = a_0 x^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} \qquad (83)$$

Step 8 Repeat the process for the second root α_2, being careful to treat the right case depending on whether $\alpha_1 - \alpha_2$ is an integer. In this case, $\alpha_1 - \alpha_2 = 1$ is an integer, so we are in Case IIb. At $n = N$, we obtain $0 * b_1 = 0$. There is no contradiction, and b_1 is arbitrary and can be set to zero (CaseIIb(ii)). Following the recursion forward with $b_0 \neq 0$, similar computations as above yield

$$y_2(x) = b_0 x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \qquad (84)$$

Step 9 The general solution is a combination of the two solutions. Thus the general solution is

$$y(x) = C_1 x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + C_2 x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

But you might recognise the series for sine and cosine here, with a root x out front! In fact, the general solution to (76) (which is Bessel's equation of order $1/2$) is

$$y(x) = C_1 \frac{\cos x}{\sqrt{x}} + C_2 \frac{\sin x}{\sqrt{x}}$$

5.4   Derivative method

Suppose α_1 is a double root of $F(\alpha)$. Let $a_0 = 1$ and solve (73) for $a_1, a_2 \ldots$ with <u>arbitrary</u> α (i.e. $F(\alpha)$ not generally $= 0$). Thus, the $a_n = a_n(\alpha)$, and we can think of α as a parameter in the series

$$y(x; \alpha) \equiv (x - x_0)^{\alpha} + \sum_{n=1}^{\infty} a_n(\alpha)(x - x_0)^{n+\alpha}$$

$$\Rightarrow Ly = L(x - x_0)^{\alpha} = (x - x_0)^{\alpha-2} F(\alpha) \qquad (85)$$

We know that $y(x; \alpha_1)$ is a solution. But α_1 is double root, which implies

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} F|_{\alpha=\alpha_1} = 0$$

The idea is to differentiate (85), then set $\alpha = \alpha_1$. Since L has no dependence on α,

$$\frac{\partial}{\partial \alpha}(Ly)|_{\alpha=\alpha_1} = 0 = L\left(\frac{\partial y}{\partial \alpha}\right)|_{\alpha=\alpha_1}$$

$$\Rightarrow y_2 = \left(\frac{\partial y}{\partial \alpha}\right)|_{\alpha=\alpha_1}$$

is also a solution. Specifically, to get a more concise form:

$$\left(\frac{\partial y}{\partial \alpha}\right)|_{\alpha=\alpha_1} = \sum_{n=0}^{\infty} a_n(\alpha_1)(x-x_0)^{n+\alpha_1} \ln(x-x_0) +$$

$$\sum_{n=0}^{\infty} \frac{\mathrm{d}a_n}{\mathrm{d}\alpha}|_{\alpha=\alpha_1}(x-x_0)^{n+\alpha_1} \tag{86}$$

$$= y_1(x)\ln(x-x_0) + \underbrace{\sum_{n=0}^{\infty} b_n(x-x_0)^{n+\alpha_1}}_{=(x-x_0)^{\alpha_1}C(x)}$$

- Derivative method can be used to determine b_n, however, a closed form for $a_n(\alpha)$ for general α is required for this! It is usually easier to just use the appropriate form of the series.
  Plug it into the equation, and compare coefficients (as you would do for a power series expansion). After plugging in, the terms containing the log terms should cancel.

6 Special functions

We have seen in the previous section a method for constructing solutions to ODEs with non-constant coefficients and singular points. For any given problem, the success of the method and the utility of the solution depends on whether one can obtain a direct formula for the series coefficients.

In this section, we explore several special functions, which occur commonly enough to have a name, and for which the "hard work", the series solution method, has already been done.

6.1 Bessel Functions

These particularly common functions can be motivated by considering the vibrating membrane of a circular drum. Let $U(x, y, t)$ be the position of the membrane at time t and position (x, y), compared to the flat horizontal rest state.

The governing equations for the membrane are (wave equation):

$$U_{tt} = c^2 \Delta U \qquad \text{for } x^2 + y^2 < 1 \qquad \text{(Newton's second law \& elastic stresses)}$$
$$U = 0 \qquad \text{at } x^2 + y^2 = 1 \qquad \text{(Membrane pinned at boundary.)}$$

Separation of variables $U(x, y, t) = v(t)u(x, y)$ yields

$$\frac{v_{tt}}{v} = \frac{\Delta u}{u} = \text{const} \equiv -\lambda,$$

$$\text{i.e.} \qquad \Delta u = -\lambda u.$$

Switch to polar coordinates: u now depends on r, θ:

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \lambda u = 0 \qquad 0 < r < 1, \quad 0 \le \theta \le 2\pi,$$

$$u = 0 \qquad r = 1, \quad 0 \le \theta \le 2\pi$$

$$u \text{ periodic in } \theta.$$

This is a PDE eigenvalue problem, we need to find λ for which there are non-trivial solutions $u(r, \theta)$.

What about $r = 0$? Note that in

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}$$

the first derivative of u has a singular coefficient. As for ODEs with singular points, we may therefore expect that this will give rise to singularities in the solutions of the PDE and thus we impose the condition

$$u \text{ bounded as } r \to 0,$$

since we expect the solution of the governing equations for the drum membrane to be bounded.

Since u is periodic in θ we can expand u into a Fouries series in θ:

$$u(r, \theta) = U_0(r) + \sum_{n=1}^{\infty} U_n(r) \cos n\theta + V_n(r) \sin n\theta;$$

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substituting this into the previous set of equations gives ($' = d/dr$)

$$\frac{1}{r}\left(rU_n'\right)' + \left(\lambda - \frac{n^2}{r^2}\right)U_n = 0, \quad \text{for } 0 \le r < 1, \tag{87a}$$

$$U_n = 0 \quad \text{at } r = 1, \tag{87b}$$

$$U_n \text{ bounded as } r \to 0. \tag{87c}$$

The same equations hold for the V_n. Now eliminate λ by rescaling: $U_n(r) = y(x)$, $x = \sqrt{\lambda}r$,

$$x^2 y'' + xy' + (x^2 - n^2)y = 0,$$

and we arrive at Bessel's equation (for integer $n \ge 0$.).

6.2 Bessel functions of first and second kind

Bessel's equation (BE) has a regular singular point at $x = 0$, with indicial equation $\alpha(\alpha - 1) + \alpha - n^2 = 0$, the solutions of which are $\alpha_1 = n$, $\alpha_2 = -n$ (double root for $n = 0$).

The general solution is given as a linear combination of two linearly independent solutions. A detailed discussion along the lines of Section 5 reveals that one is locally given at $x = 0$ by a Frobenius series with the exponent $\alpha_1 = n$ and the other by a Frobenius series with exponent $\alpha_2 = -n$ plus $\ln(x)$ times the first solution (case II(b)(i) in our general discussion of ODEs with singular points).

The first Frobenius series with a specific normalization of the leading coefficient of the expansion defines the *Bessel functions of first kind*

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k}$$

for integer $n \ge 0$.
Similarly, a specifically normalized choice for the second expansion defines the *Bessel functions of second kind*

$$Y_n(x) = \frac{2}{\pi} \ln(x/2) J_n(x) - \frac{(x/2)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(x^2/4\right)^k$$

$$- \frac{(x/2)^n}{\pi} \sum_{k=0}^{\infty} [\psi(k+1) + \psi(n+k+1)] \frac{\left(-x^2/4\right)^k}{k!(n+k)!},$$

where $\psi(m) = -\gamma + \sum_{k=1}^{m-1} k^{-1}$, $(m \geq 1)$ and $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. (Details regarding these expansions will be studied in the problem sets on problem sheet 7.)

Some Bessel function fun facts:

- Since Bessel's equation has only one singular point for finite x, the series for J_n and in Y_n have an infinite radius of convergence.

- Also, J_n and Y_n are oscillating functions that decay as $x \to \infty$. They have an infinitude of discrete zeros for $x \geq 0$, which are quite important and have therefore been tabulated (for example in Abramowitz and Stegun).

- At $x = 0$, the behaviour of the two kinds of Bessel functions is quite different. For J_n, we have $J_n(0) = 0$ if $n > 0$, and $J_0(0) = 1$, while $Y_n \to \infty$ as $x \to 0$.

- Two recursion relations, which can be derived from the local expansions:

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x),$$
$$J_{n+1}(x) = -2J_n'(x) + J_{n-1}(x).$$

  The same relations also hold for the Y_n's. In principle, these relations can be used to compute the Bessel functions, however, the straightforward way – calculating the values for larger n from those for smaller n – is usually numerically unstable and is therefore not recommended.

Many more relations as well as theory exist. A vast collection of results for the Bessel functions can be found in particular in Abramowitz and Stegun; some derivations in Riley et al. But for now, back to the vibrating drum. We can now express the general solution for (87a) in terms of Bessel functions as

$$U_n(r) = aJ_n\left(\sqrt{\lambda}r\right) + bY_n\left(\sqrt{\lambda}r\right).$$

The boundedness condition (87c) requires $b = 0$. A non-trivial solution therefore requires that we set $a \neq 0$, without loss of generality: $a = 1$. Thus, the boundary condition (87b) at $r = 1$ leads to

$$J_n\left(\sqrt{\lambda}\right) = 0,$$

i.e. $\sqrt{\lambda}$ has to be one of the zeros of J_n. If we label the zeros of J_n by α_m $m = 1, 2, \ldots$, (sorted, for example, in ascending order), then the eigenvalues for (87) are

$$\lambda = \alpha_m^2 \qquad m = 1, 2, \ldots$$

with corresponding eigenfunctions $J_n(\alpha_m r)$, $m = 1, 2, \ldots$. (Note that n is kept fixed, and that the zeros depend on n!)

The differential equation (87a) can be written in Sturm-Liouville form by multiplying through with r; i.e. (87) is a singular SL problem with weighting function r, and we therefore have the following orthogonality relations between the eigenfunctions

$$\int_0^1 r J_n(\alpha_l r) J_n(\alpha_m r) \, dr = 0 \qquad \text{for } l \neq m;$$

a separate calculation is required for $l = m$:

$$\int_0^1 r J_n^2(\alpha_m r) \, dr = \frac{1}{2} \left(J_n'(\alpha) \right)^2.$$

6.3 Legendre equation and Legendre functions

Legendre equations/functions arise from studying eigenvalue problems for the 3D Laplace operator in spherical coordinates.

The *associated Legendre equation* is given by

$$(1 - x^2)y'' - 2xy' + \left(l(l+1) - \frac{m^2}{1 - x^2} \right) y = 0,$$

or in self-adjoint form

$$\left((1 - x^2)y' \right)' + \left(l(l+1) - \frac{m^2}{1 - x^2} \right) y = 0.$$

The numbers m and l can in general be complex; here, we will focus on the case where m and l are non-negative integers. The solutions of the associate Legendre equation are the *associated Legendre functions* and are denoted by P_l^m; for $m = 0$, we drop the 'associated' and speak of the Legendre equation and functions, usually denoted by P_l.

6.3.1 Properties

1. The points $x = \pm 1$ and $x = \infty$ are regular singular points of the associated Legendre equation. The indicial exponents for $x = 1$ are $-m/2$ and $m/2$. Thus, the local expansion yields one bounded and one unbounded solution at $x = 1$. The same is true for $x = -1$.

2. If we replace $l(l+1)$ in the self-adjoint form of the associated Legendre equation by λ,

$$\left((1 - x^2)y'\right)' + \left(-\frac{m^2}{1 - x^2}\right)y + \lambda y = 0, \tag{88}$$

and consider bounded solutions on $-1 < x < 1$, we see that boundedness imposes two conditions, one at each end of the interval. This suggests that (88) is a singular Sturm-Liouville problem (with coefficient functions $p = 1 - x^2$, $q = -m^2/(1 - x^2)$, $r = 1$) with discrete eigenvalues. Indeed, the eigenvalues are exactly of the form $\lambda = l(l+1)$ with integer $l \geq m$. The eigenfunctions are the corresponding associated Legendre functions, i.e. P_l^m. From Sturm-Liouville theory, we infer the orthogonality relation

$$\int_{-1}^{1} P_l^m(x) P_n^m(x)dx = 0 \qquad \text{for } l \neq n.$$

The case $l = n$ requires explicit calculation, see problem sheet.

3. For $\underline{m = 0}$ the Legendre functions (without 'associated') are polynomials, and are given explictly by a so-called *Rodrigues' formula*:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}\left[(x^2 - 1)^l\right].$$

Of course, to find the general solution of the Legendre equation we need a second, linearly independent solution, and this is given by the Legendre function of 2nd kind, denoted by Q_n. These solutions are unbounded at $x = \pm 1$. For the case $n = 0$, the solution Q_0 was stated on problem sheet 4:

$$Q_0(x) = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right).$$

4. For the general case, $0 \leq m \leq l$, the associated Legendre functions of first and second kind are given by

$$P_l^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m P_l}{dx^m}$$

and

$$Q_l^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m Q_l}{dx^m}$$

respectively. The associated Legendre functions P_l^m are polynomials, if and only if m is even.

5. There are several recurrence relations for (associated) LF, e.g.: These relations, and further properties can be found in Abramowitz and Stegun, and some derivations in Riley at al.

6.4 Generalisation: Orthogonal Polynomials

There are other second order linear ODEs with families of orthogonal polynomials as solutions. Often, the orthogonality relations

$$\int_a^b p_m(x) p_n(x) \omega(x) dx = 0 \qquad m \neq n$$

with a fixed weighting function $\omega(x)$ (in general non-trivial, i.e. $\not\equiv 1$) can by inferred by formulating appropriate Sturm-Liouville (eigen)problems. Orthogonal polynomials play an important role in approximation theory, for the construction of numerical methods to discretized differential equations, and in many applications, e.g. from physics.

One can in fact give a complete classification of all infinite families of orthogonal polynomials that can arise from second order linear differential equations (we omit here some specific conditions that are needed to make this a precise statement). The most important ones include (n is a non-negative integer):

1. The "Jacobi-like" polynomials, to which the Legendre, the Chebychev, and the Gegenbauer polynomials belong. These arise from DEs of the type
$$x(1 - x)y'' + (a + bx)y' + \lambda y = 0,$$
with constants a and b and an appropriate discrete set of λ.

2. The Laguerre and associated Laguerre polynomials, which are solutions of

$$xy'' + (k + 1 - x)y' + \lambda y = 0,$$

for $k \neq -1, -2, \ldots$ and an appropriate discrete set of λ. (For the Laguerre polynomials, i.e. without 'associated', we have $k = 0$, and $\lambda = n$.)

3. Hermite polynomials, which are solutions of the Hermite equation

$$y'' - 2xy' + \lambda y = 0 \qquad \text{with } \lambda = 2n.$$

These families share many similar structural properties, e.g. they are given explicitly by Rodrigues' formulae and have similar recurrence relations, see Abramowitz and Stegun.

7 Approximation techniques: Asymptotic Expansions and Regular Perturbations

A complex mathematical problem often cannot be solved exactly but it may contain parameters that represent physical constants or other conditions relevant to particular circumstances. If these parameters are small or large it may be possible to exploit this in order to derive approximate solutions to the original problem. Doing this in a systematic manner is possible through the use of asymptotic expansions. In this section a basic framework for using these methods is presented and some insight given into how to derive approximations. Such methods can be put on a more rigorous footing but we shall not pursue this further here.

We shall consider mathematical problems that have been put into a form, usually through a process of "nondimensionalisation" that allows simple parameters to be considered in the problem. More details on how to nondimensionalise a given physical problem can be found elsewhere and in Part B and C applied mathematical courses.

7.1 Definitions: order notation and twiddles

To start it is necessary to give a basic structure where approximations to a function can be considered as some parameter in the function becomes large or small.

The goal of the next few definitions is to capture the idea that a function is 'the same size as' or 'much bigger (smaller)' than another near a point

We consider two functions as x tends towards a particular value x_0. Let $f(x), g(x) \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ (often $x_0 = 0$ or ∞ so it will be a small or large parameter).

There are now some definitions that will allow the size of functions to be compared. The first is close to the idea that one function is "less than or equal" to the magnitude of another function.

Definition ('Big O' notation). We write

$$f(x) = \mathrm{O}\left(g(x)\right) \quad \text{as } x \to x_0$$

if $\exists A > 0$ such that $|f| < A\,|g|$ for all x near x_0 (compare to definition of limit, Lipschitz).

Examples.
$$\sin(2x) = \mathrm{O}\left(x\right) \ \text{as } x \to 0$$
$$3x + x^3 = \mathrm{O}\left(x\right) \quad \text{as } x \to 0$$
$$5x^2 + x^{-3} - \mathrm{e}^{-x} = \mathrm{O}\left(x^2\right) \quad \text{as } x \to \infty.$$

The next definition defines when two functions are "equal" (this includes any coefficients)

Definition ('Twiddles' or 'is asymptotic to'). We write

$$f(x) \sim g(x) \quad \text{if } \frac{f}{g} \to 1 \text{ as } x \to x_0$$

("f is asymptotic to g as $x \to x_0$").

Examples.
$$\sin 2x \sim 2x \quad \text{as } x \to 0$$
$$x + \mathrm{e}^{-x} \sim x \quad \text{as } x \to \infty$$

and finally we can consider one function to be strictly greater in magnitude than another function

Definition ('little o' notation). We write

$$f(x) = \mathrm{o}\left(g(x)\right) \quad \text{as } x \to x_0$$

if $\lim_{x \to x_0} \frac{f}{g} = 0$.

(Note that this relationship is also written in the form "$f \ll g$ as $x \to x_0$").

Examples.

$$9x^2 - 4x^5 = \mathrm{o}\,(x) \qquad \text{as } x \to 0$$

$$\frac{3}{x^2} - 3\mathrm{e}^{-x} = \mathrm{o}\left(\frac{1}{x}\right) \qquad \text{as } x \to \infty$$

.

Note that whenever using the notation '$O()$', '$\sim$' or '$o()$' you should include in the statement what value x is tending to.

Example. You know from previous courses Taylor's theorem for a smooth function $f(x)$. This allows the statements to be made:

$$
\begin{aligned}
f(x) &= f(x_0) + (x - x_0)f'(x_0) + \mathrm{O}\left((x - x_0)^2\right) \qquad \text{as } x \to x_0 \\
f(x) &= f(x_0) + (x - x_0)f'(x_0) + \mathrm{o}\left((x - x_0)\right) \qquad \text{as } x \to x_0 \\
f(x) &= f(x_0) + (x - x_0)f'(x_0) + \mathrm{o}\left((x - x_0)^{3/2}\right) \qquad \text{as } x \to x_0 \\
f(x) &\sim f(x_0) \qquad \text{as } x \to x_0 \\
f(x) - f(x_0) &\sim (x - x_0)f'(x_0) \qquad \text{as } x \to x_0.
\end{aligned}
$$

7.2 Asymptotic Sequences

We now use these ideas of determining the magnitude of a function to examine how to solve problems approximately. We shall change the notation here a bit to allow us to use the ideas in many different situations.

In the previous work x was considered as the parameter that was tending to a particular value. In practice, as mentioned earlier, some physically relevant parameter in a problem will be considered to be large or small. To focus these ideas in the following work consider the physical parameter to be ϵ and take the particular case where the solution to a problem is sought in the case where ϵ is small so that we take $\epsilon \to 0$. (as this is generally understood from context, we will not always repeat the phrase 'as $\epsilon \to 0$').
To start the approximation of a function it is useful to generalise the idea of a Taylor series for a function by expanding the function in terms of ϵ. To do so, a sequence is needed similar to the powers of $x - x_0$ in Taylor series.

Definition. A set of functions $\{\phi_n(\epsilon)\}_{n=0,1,2,\ldots}$ is an *asymptotic sequence* as $\epsilon \to 0$ if, for each $i > 0$, $\phi_{n+i}(\epsilon) = \mathrm{o}\,(\phi_n(\epsilon))$ as $\epsilon \to 0$, i.e. each subsequent term in the sequence is of smaller magnitude than the previous term.

Examples.

$$\{1, \epsilon, \epsilon^2, \epsilon^3, \ldots\}$$

$$\{1, \epsilon^{1/2}, \epsilon, \epsilon^{3/2}, \ldots\}$$
$$\{1, \epsilon, \epsilon \log \epsilon, \epsilon, \epsilon^2 \log \epsilon, \ldots\}$$

Note the use of "..." to indicate the sequence continues.

Definition. A function $f(\epsilon)$ has an *asymptotic expansion* with respect to $\{\phi_n(\epsilon)\}$ as $\epsilon \to 0$ if there exist constants a_n such that

$$f(\epsilon) \sim \sum_n a_n \phi_n(\epsilon).$$

This means that

$$f(\epsilon) = \sum_{n=0}^{N} a_n \phi_n(\epsilon) + \mathrm{o}\left(\phi_N(\epsilon)\right) \quad \forall N.$$

From earlier you know that if $f(x)$ is smooth then the coefficients of a Taylor expansion are unique; a similar property holds for an asymptotic expansion.

Lemma. *Given $\{\phi_n\}$, the coefficients a_n are unique.*

Proof. By induction on n — try it at home. $\qquad\qquad\qquad\qquad$ □

Note. The function defines the expansion but not vice versa.

Example. If $\phi_n(\epsilon) = \epsilon^n$, $n = 0, 1, 2, \ldots$, then

$$\frac{1}{1-\epsilon} \sim 1 + \epsilon + \epsilon^2 + \cdots \quad \text{as } \epsilon \to 0$$

but

$$\frac{1}{1-\epsilon} + \mathrm{e}^{-1/\epsilon} \sim 1 + \epsilon + \epsilon^2 + \cdots \quad \text{as } \epsilon \to 0 \quad \text{as well}$$

because

$$\mathrm{e}^{-1/\epsilon} = \mathrm{o}\left(\epsilon^n\right) \quad \forall n$$

("exponentially small" ("transcendentally small"))

Hence asymptotic expansions generalise the concept of Taylor series but lose the property of defining the function. In addition such asymptotic expansions may, and in most cases do, have zero radius of convergence but we do not consider such issues here.

In solving a physical problem where ϵ is a parameter (e.g. a coefficient) the series may look like

$$f(\vec{x}, \epsilon) \sim \sum_n a_n(\vec{x}) \phi_n(\epsilon)$$

and we shall only seek the first few terms of the expansion as the approximation of the function.

7.3   Approximate roots of equations, dominant balance

To start using asymptotic methods consider the problem of finding the roots of equations. Here the roots are given by values of x and the parameter in the problem is ϵ, and the roots are sought in the case $\epsilon \to 0$. Hence the roots are $x(\epsilon)$. To focus ideas, consider a simple case where the exact roots can be easily found.

Example. Solve approximately the quadratic

$$\epsilon x^2 + x - 1 = 0 \qquad \text{as } \epsilon \to 0.$$

Note. It is reasonable to consider that the solution "as $\epsilon \to 0$" might be found by taking $\epsilon = 0$ so that

$$x - 1 = 0 \qquad \Rightarrow \qquad x = 1$$

is the solution. But the original problem was a *quadratic*, so we will use asymptotics to see how we get a better approximation to this root near $x = 1$ but also how to find an approximation to the other root.

- *First try*: The simplest strategy is to guess that the asymptotic expansion of $x(\epsilon)$ is simply a Taylor series, so we guess

$$x(\epsilon) \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

where x_0, x_1, x_2 etc are constants (they do not depend on ϵ).
Putting this guess into the quadratic gives

$$\epsilon(x_0 + \epsilon x_1 + \cdots)^2 + (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) - 1 = 0$$

and collecting together the different terms in the asymptotic sequence then gives

$$(x_0 - 1) + \epsilon(x_0^2 + x_1) + \epsilon^2(2x_0 x_1 + x_2) + \mathrm{O}\left(\epsilon^3\right) = 0.$$

Then since this is true *for all ϵ*, and we have assumed that $x_0, x_1, \ldots$ *are independent of ϵ*, we conclude that equality must hold independently for each power of ϵ. Hence, we have

$$
\begin{array}{llllll}
\mathrm{O}\,(1) & : & x_0 - 1 & = & 0 \Rightarrow \quad x_0 & = \quad 1 \\
\mathrm{O}\,(\epsilon) & : & x_0^2 + x_1 & = & 0 \Rightarrow \quad x_1 & = \quad -x_0^2 = -1 \\
\mathrm{O}\left(\epsilon^2\right) & : & 2x_0 x_1 + x_2 & = & 0 \Rightarrow \quad x_2 & = \quad -2x_0 x_1 = 2 \\
\cdots & & \text{etc.} & & &
\end{array}
$$

Hence we find we can systematically improve the approximation of the root near $x = 1$ and it is

$$x \sim 1 - \epsilon + 2\epsilon^2 + \mathrm{O}\left(\epsilon^3\right) \qquad \text{as } \epsilon \to 0$$

This root near $x = 1$ has been found from considering a 'dominant balance' between x and 1 in the equation (Notice there are three terms in the equation and this balance leaves a small error in the equations):

$$\underbrace{\epsilon x^2}_{\text{small}} + \underbrace{x - 1}_{\text{balance (nearly, } \epsilon x^2 \text{ gives connections)}} = 0$$

If we want to approximate the other root, we need to consider balances between different terms in the equation,

- *Second try* Let's consider the case where we try balancing $\epsilon^2 x$ and 1 (This is the dominant balance only if the other term in the equation is small, i.e. $x \ll (\epsilon^2 x, 1)$).

- To get a balance between ϵx^2 and 1 requires that $\Rightarrow x = \mathrm{O}\left(\frac{1}{\sqrt{\epsilon}}\right)$. This gives the following sizes for the terms:

$$\underbrace{\epsilon x^2}_{\mathrm{O}(\epsilon) \times \mathrm{O}\left(\frac{1}{\epsilon}\right) = \mathrm{O}(1)} + \underbrace{x}_{\mathrm{O}\left(\frac{1}{\sqrt{\epsilon}}\right)} - \underbrace{1}_{\mathrm{O}(1)} = 0$$

but this contradicts the assumption that x was smaller than the terms ϵx^2 and 1 (if we continued on this route we'd also get *two* more roots because of the $\pm$ and hence that also seems wrong)

We conclude that $x \not\ll (\epsilon x^2, 1)$ and this balance does not make sense

- *Third try* There is one other possible balance left to try namely: $\epsilon x^2 \sim x$. Of course this will only be a dominant balance by assuming that $1 \ll (\epsilon x^2, x)$. To get this balance we require $x \sim \frac{1}{\epsilon}$ and when we check we find $x \sim \frac{1}{\epsilon} \gg 1$ as $\epsilon \to 0$ so this is a consistent balance (aka, a "dominant" balance)

Now, use this new balance to find an approximate solution. To get the balance we need to take $x = \frac{1}{\epsilon} y$ (where we assume $y = \mathrm{O}(1)$).

$$\Rightarrow \frac{\epsilon y^2}{\epsilon^2} + \frac{y}{\epsilon} - 1 = 0 \Rightarrow y^2 + y - \epsilon = 0$$

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Again assume this simplest possible asymptotic expansion for the unknown y by taking $y \sim y_0 + \epsilon y_1 + \cdots$. Putting this into the equation gives

$$(y_0 + \epsilon y_1 + \cdots)^2 + (y_0 + \epsilon y_1 + \cdots) - \epsilon = 0.$$

Then, as before, equate powers of ϵ to find:

$$
\begin{aligned}
\mathrm{O}\,(1) \quad &: \quad y_0^2 + y_0 &= \ 0 \ &\Rightarrow \ y_0 \ = \ 0, -1 \\
\mathrm{O}\,(\epsilon) \quad &: \quad 2y_0 y_1 + y_1 - 1 &= \ 0 \ &\Rightarrow \ y_1 \ = \ 1, \underbrace{\frac{1}{2y_0 + 1}}_{-1} \\
\cdots \quad & \qquad y_1(2y_0 + 1) = 1
\end{aligned}
$$

(Note $y_0 = 0$ actually corresponds to the previous root we found with the original dominant balance)

$$
\Rightarrow \quad
\begin{cases}
y \ = \quad 0 + \epsilon + \mathrm{O}\,(\epsilon)^2 \quad \Rightarrow \quad x \ = \quad \underbrace{1 + \mathrm{O}\,(\epsilon)}_{\text{root from before}} \\[2mm]
y \ = \ -1 - \epsilon + \mathrm{O}\,(\epsilon^2) \quad \Rightarrow \quad x \ = \ -\dfrac{1}{\epsilon} - 1 + \mathrm{O}\,(\epsilon)
\end{cases}
$$

We didn't find the second value of x before because we had tacitly assumed that the largest term was of size $\mathrm{O}\,(1)$.

Note for algebraic equations to check all possibilities, you can do dominant balance to help determine which expansions you need.

Now consider using the same ideas to find the roots of an equation where you cannot find the exact solution.

Example. Find the asymptotic expansion of all the roots of

$$x\mathrm{e}^{-x} = \epsilon \qquad \text{as } \epsilon \to 0.$$

Determine the number of roots and their approximate location
A sketch can help determine what roots we need to find. For example write the equation as

$$x = \epsilon \mathrm{e}^x$$

and by plotting the functions on either side of the equality we can see that if $\epsilon \ll 1$ there are two points where the two graphs intersect. Hence there are two roots for sufficiently small ϵ; one small root and one large root
Exercise: show that there exist two roots if $\epsilon < \mathrm{e}^{-1}$.

Small root: If x is small then to get a dominant balance we have that $e^x \sim 1$ so the equation is approximately $x \sim \epsilon 1$ so this balance requires $x \sim \epsilon$. Using this a reasonable assumption is that a suitable asymptotic expansion for this root is $x \sim \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots$. Using this gives

$$(\epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots) = \epsilon e^{\epsilon x_1 + \epsilon^2 x_2 + \cdots}$$

$$= \epsilon(1 + \epsilon x_1 + \epsilon^2 x_2 + \frac{1}{2}\epsilon^2 x_1^2 + \cdots) \quad \text{(Taylor series)}$$

- equate like powers of ϵ:

$$
\begin{array}{llll}
O\left(\epsilon\right) & : & \Rightarrow \ x_1 & = \ 1 \\
O\left(\epsilon^2\right) & : & \Rightarrow \ x_2 & = \ 1 \\
O\left(\epsilon^3\right) & : & \Rightarrow \ x_3 & = \ x_2 + \frac{1}{2}x_1^2 = \frac{3}{2} \\
\cdots
\end{array}
$$

Large root:
It is less obvious how large this root is or what asymptotic series should be assumed. Hence we write the solution in a general form as

$$x \sim x_0\phi_0(\epsilon) + x_1\phi_1(\epsilon) + x_2\phi_2(\epsilon) + \cdots \qquad \text{where } \phi_0 \gg \phi_1 \gg \cdots \quad \epsilon \to 0$$

To find the other possible dominant balance, take logs of both sides of $xe^{-x} = \epsilon$ to get:

$$\log x - x = \log \epsilon$$

Then note that, as $x \to \infty$, $x \gg \log x$, so that the dominant term on the LHS is $-x$. and the balance gives $-x \sim \ln \epsilon$. Now plug the approximation $x \sim x_0\phi_0 + x_1\phi_1 + \cdots$ in the equations so that

$$\log(x_0\phi_0 + x_1\phi_1 + x_2\phi_2 + \cdots) - (x_0\phi_0 + x_1\phi_1 + x_2\phi_2 + \cdots) = \log \epsilon$$

The dominant terms are $-x_0\phi_0$ and $\log \epsilon$ (all others are smaller), so take $x_0\phi_0 = -\log \epsilon$ (ie: $x_0 = 1$ and $\phi_0 = -\log \epsilon$ so that ϕ_0 is positive!)
Now expand the ln term by noting that we can write

$$\log(x_0\phi_0 + x_1\phi_1 + \cdots) = \log x_0\phi_0 + \log\left(1 + \frac{x_1\phi_1}{x_0\phi_0} + \cdots\right)$$

$$\sim \log x_0\phi_0 + \frac{x_1\phi_1}{x_0\phi_0} + \cdots$$

Returning to the equations this gives

$$\underbrace{\log x_0\phi_0}_{\text{largest remaining term}} + \underbrace{\frac{x_1\phi_1}{x_0\phi_0}}_{\text{2nd largest remaining term}} + \cdots - (\cancel{x_0\phi_0} + x_1\phi_1 + \cdots) = \cancel{\log \epsilon}$$

Separating the terms by the size in the asymptotic series

$$
\begin{cases}
x_1\phi_1 = \log x_0\phi_0 = \log|\log\epsilon| \\
x_2\phi_2 = \dfrac{x_1\phi_1}{x_0\phi_0} = \dfrac{\log|\log\epsilon|}{|\log\epsilon|} \\
\cdots
\end{cases}
$$

Hence the solution can be written

$$
x \sim x_0\phi_0(\epsilon) + x_1\phi_1(\epsilon) + x_2\phi_2(\epsilon) + \cdots
$$

where

$$
x_0 = 1 \qquad \phi_0 = |\log\epsilon|
$$
$$
x_1 = 1 \qquad \phi_1 = \log|\log\epsilon|
$$
$$
x_2 = 1 \qquad \phi_2 = \frac{\log|\log\epsilon|}{|\log\epsilon|}
$$

For this large root it was not obvious what the form of the asymptotic sequence should be. Hence rather than assuming the form of the asymptotic sequence, $\phi_n(\epsilon)$ these were found as each of the terms was found.

7.4   Regular Perturbations in ODEs

Having shown how to use asymptotic methods to find approximate roots to asymptotic and transcendental equations it is now useful to see how the same ideas apply to solving ODEs. The simplest case is when the solution to an ODE is a regular expansion.

A *regular expansion* using an asymptotic sequence $\{\phi_n(\epsilon)\}$ is an asymptotic expansion in which we don't rescale any of the independent variables (this is the same procedure used for the root $x = 1 - \epsilon + \cdots$ in the earlier quadratic equation). It is easiest to see how this works via an example.

Example. Find the solution u of the following problem as $\epsilon \to 0$

$$
\begin{cases}
u'' = 1 + \epsilon \sin u, & 0 < x < 1 \\
u(0) = u(1) = 0
\end{cases}
$$

Note. This can be solved exactly, so you can use the exact solution to check your asymptotics.

The solution $u(x, \epsilon)$ depends on both x and ϵ. As a simplest assumption take the solution to be approximated by the asymptotic expansion

$$u(x, \epsilon) \sim u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots$$

Putting this into the ODE gives

$$
\begin{aligned}
u_0'' + \epsilon u_1'' + \cdots &= 1 + \epsilon \sin(u_0 + \epsilon u_1 + \cdots) \\
&\sim 1 + \epsilon(\sin u_0 + \epsilon u_1 \cos u_0 + \cdots)
\end{aligned}
$$

with

$$u(0, \epsilon) \sim u_0(0) + \epsilon u_1(0) + \cdots = 0$$
$$u(1, \epsilon) \sim u_0(1) + \epsilon u_1(1) + \cdots = 0$$

and then equating each power of ϵ gives

- O (1) : $u_0'' = 1$ with $u_0(0) = u_0(1) = 0$

$$\Rightarrow u_0 = -\frac{1}{2}x(1 - x)$$

- O (ϵ): $u_1'' = \sin u_0 = -\sin\left[\frac{1}{2}x(1 - x)\right]$ with $u_1(0) = u_1(1) = 0$

$$\Rightarrow u_1 = -\int_0^x (x - t) \sin\left[\frac{1}{2}t(1 - t)\right] dt + Ax + B,$$

where the boundary conditions give $B = 0$ (and A is found from $u_1(0) = 0$).

Example. Small oscillations for a pendulum
Consider solving the problem

$$\ddot{\theta} + \sin \theta = 0, \quad \theta(0) = \epsilon, \quad \dot{\theta}(0) = 0 \qquad \epsilon \to 0$$

The simplest possible assumption to take for the asymptotic expansion of $\theta(t, \epsilon)$ is to take a Taylor series

$$\theta \sim \theta_0 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \cdots.$$

A simple calculation shows that $\theta_0 = 0$ (though you can certainly check this), so the expansion is taken as

$$\theta \sim \epsilon \theta_1 + \epsilon^2 \theta_2 + \cdots.$$

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Plugging this into the problem gives

$$(\epsilon\ddot{\theta} + \epsilon^2\ddot{\theta}_2 + \cdots) + \sin(\epsilon\theta_1 + \epsilon^2\theta_2 + \cdots) = 0$$

with

$$\begin{cases} \theta_1(0) &=& 1 \\ \dot{\theta}_1(0) &=& 0 \end{cases} \quad \begin{cases} \theta_2(0) &=& \theta_3(0) &=& \cdots &=& 0 \\ \dot{\theta}_2(0) &=& \dot{\theta}_3(0) &=& \cdots &=& 0 \end{cases} .$$

We then note that the trigonometric function can be expanded using

$$\sin(\epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 + \cdots) \sim \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 - \frac{1}{6}\epsilon^3\theta_1^3 - o\left(\epsilon^3\right)$$

so that the ODE becomes

$$\epsilon\ddot{\theta}_1 + \epsilon^2\ddot{\theta}_2 + \epsilon^3\ddot{\theta}_3 + \cdots + \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 - \frac{1}{6}\epsilon^3\theta_1^3 + \cdots = 0 .$$

We now consider equating powers of ϵ in the ODE and in the boundary conditions. Considering each size of terms we find

$$\ddot{\theta}_1 + \theta_1 = 0 \quad \Rightarrow \theta_1 = \cos t \quad \text{(using BCs)}$$

$$\ddot{\theta}_2 + \theta_2 = 0 \quad \Rightarrow \theta_2 \equiv 0$$

$$\begin{aligned} \ddot{\theta}_3 + \theta_3 &=& \frac{1}{6}\cos^3 t \\ &=& \frac{1}{24}(3\cos t + \cos 3t) \end{aligned}$$

(Note it is easiest to solve this by expanding in component harmonics)

$$\theta_3 = \frac{1}{16}t\sin t - \frac{1}{192}(\cos 3t - \cos t)$$

The asymptotic expansion of the solution to the ODE is therefore

$$\theta \sim \epsilon\cos t + \epsilon^3\left(\frac{1}{16}t\sin t - \frac{1}{192}(\cos 3t - \cos t)\right) + \ldots$$

Note. This solution is a good approximation to the solution at least initially. However, we would hope that the solution might be valid as we take t to be large and a difficulty arises. The term $t\sin t$ in θ_3 — called a *secular term* — grows unboundedly as $t \to \infty$. In particular this means that the assumption about the asymptotic expansion for θ will become invalid if t is very large. We have assumed that

$$\epsilon\theta_1 = \epsilon\cos t = o(\epsilon^3\theta_3) = o(\epsilon^3 t\sin t) .$$

However, this become invalid (as $\epsilon \to 0$) when $t = O(1/\epsilon^2)$.

To approximate the solutions for larger times, beyond this limit, it is necessary to use *multiscale perturbation theory* (but we shall not discuss this here and leave it for later courses such as C6.3a)

Example (Another example where taking an infinite interval for the independent variable leads to trouble). Consider solving

$$u' = u + \epsilon u^2, \quad u(0) = 1, \quad x > 0, \quad \epsilon \to 0$$

and make the simple assumption about the asymptotic expansion

$$u \sim u_0 + \epsilon u_1 + \cdots.$$

and equating powers of ϵ then gives us

$$u_0 = \mathrm{e}^x$$

$$u_1' = u_1 + \mathrm{e}^{2x}, u_1(0) = 0 \quad \Rightarrow u_1 = \mathrm{e}^{2x} - \mathrm{e}^x.$$

If the assumption about the asymptotic expansion is now checked then the first and second terms become the same size, $\epsilon u_1 \sim u_0$, when $\epsilon \mathrm{e}^{2x} \sim \mathrm{e}^x$, i.e. when $x \sim |\log \epsilon|$.

In fact, the exact solution is

$$u = \frac{\mathrm{e}^x}{1 + \epsilon - \epsilon \mathrm{e}^x}$$

which ceases to exist when

$$x = \left| \log\left(\frac{\epsilon}{1 + \epsilon}\right) \right|.$$

7.5   Boundary Layers

The previous solutions by asymptotic expansions assumed regular expansions. However, the last examples indicated that sometimes the expansions need to be carefully considered in different regions of the independent variable. The size of these regions depends on the limiting parameter ϵ and arise in many situations. The following discusses some initial ideas on how to analyse such problems.

Suppose we try to solve

$$\epsilon y' + y = \mathrm{e}^{-x}, \quad x > 0, \quad y(0) = 0$$

If the asymptotic expansion is assumed to be $y \sim y_0 + \epsilon y_1 + \cdots$, then

$$\epsilon(y_0' + \epsilon y_1' + \cdots) + (y_0 + \epsilon y_1' + \cdots) = e^{-x}.$$

Equating the powers of ϵ then gives

$$
\begin{aligned}
y_0 &= e^{-x} && \text{with } y_0(0) = 0 \\
y_1 &= -y_0' = e^{-x} && \text{with } y_1(0) = 0 \\
&\cdots
\end{aligned}
$$

The *problem* is that we can never satisfy the boundary condition $y(0) = 0$
To examine why this problem occurs note that the *exact solution* is

$$y = \frac{e^{-x}}{1 - \epsilon} - \frac{e^{-x/\epsilon}}{1 - \epsilon}$$

Plot this function and you will find it looks like $y = e^{-x}$. However, if you look carefully at such a plot (use a very small value of ϵ) you will find that there is a nearly vertical line near $x = 0$ where the solution jumps rapidly from $y = 0$ to the graph that looks like e^{-x}
Such a rapid variation in the solution is called a "boundary layer". It is occurs in a very narrow region and examining the exact solution we can see that the behaviour is caused by the part of the solution containing $e^{-x/\epsilon}$. Hence we expect that this "boundary layer" will be approximately $x = \mathrm{O}(\epsilon)$ since this is the scale of x that this function changes over. We shall now how to include such boundary layers in our solution method.[11]
We need to change our solution method because the exact solution indicates that the derivative is large, $\mathrm{O}(1/\epsilon)$, near $x = 0$, so that $\epsilon y'$ comes back in at leading order (in the dominant balance) near $x = 0$.
The method we shall adopt to allow for this behaviour is to construct different asymptotic expressions in different regions of x and then join them up by *matching*. The conventional notation is to call the region where $x = O(1)$ the "outer region" and the very narrow layer near $x = 0$ the "inner region" (or inner layer). The expansions in the different regions (each with different scalings for x) should smoothly match together.

7.5.1   Inner and Outer Expansions

To get the ideas clear consider the case where the exact solution to the problem is known and we want to find the inner and outer expansions.

---

[11]The terminology, boundary layer, comes from Prandtl's work on fluid mechanics related to viscous flow near a boundary.

The outer expansion

Consider

$$f(x, \epsilon) = \frac{e^{-x} - e^{-x/\epsilon}}{1 - \epsilon}$$

an assuming that the region is $x = O(1)$ then this gives a regular expansion

$$f(x, \epsilon) \sim f_0 + \epsilon f_1 + \cdots \sim e^{-x}(1 + \epsilon + \epsilon^2 + \cdots)$$

and the term $e^{-x/\epsilon}$ is left out because, for $x = O(1)$, $e^{-x/\epsilon} = o(\epsilon^n)$ for all n; $f_0 + \epsilon f_1 + \cdots$

The inner expansion

We can see from the exact solution that something interesting happens to the expansion of the exponential when x is the size of ϵ ($e^{-x/\epsilon} = O(1)$ is *not* small). Hence consider the inner expansion to occur when $x = O(\epsilon)$. To allow this to occur we *rescale* the independent variable by taking $x = \epsilon y$ and assuming $y = O(1)$, so that the expansion becomes

$$\begin{aligned}
f(x, \epsilon) = F(y, \epsilon) \quad &= \quad \frac{e^{-\epsilon y} - e^{-y}}{1 - \epsilon} \\
&\sim \quad \left(1 - \epsilon y + \frac{\epsilon^2 y^2}{2} + \cdots - e^{-y}\right)(1 + \epsilon + \epsilon^2 + \cdots)
\end{aligned}$$

$$\Rightarrow F \sim F_0 + \epsilon F_1 + \cdots \sim 1 - e^{-y} + \epsilon(1 - e^{-y} - y) + O(\epsilon^2).$$

This is the *inner expansion*, it is in terms of the scaled variable $y = \frac{x}{\epsilon}$

(In general, an outer expansion has one scaling and an inner expansion has another scaling; one could imagine a problem with lots of different regions with different scalings but we do not consider these here)

7.5.2 Matching

In the previous section we showed how to create different asymptotic expansions of a single function in different regions. IF we know the function then we will find that the two approximations smoothly join together but, when we try to solve ODEs, we will need to find out how to make this smooth connection. This method of smoothly connecting two asymptotic expansions in different regions is called *matching*.

We start by considering the simple function we expanded earlier.

At leading order $(O(1))$, the outer expansion is

$$f \sim f_0 + O(\epsilon) \qquad \text{where } f_0 = e^{-x}$$

and the inner expansion is

$$F = F_0 + \mathrm{O}\,(\epsilon) \qquad \text{where } F_0 = 1 - \epsilon^{-y}$$

Recall: when $y = \mathrm{O}\,(1)$, $x = \mathrm{O}\,(\epsilon)$ and when $x = \mathrm{O}\,(1)$, $y = \mathrm{O}\,(1/\epsilon)$.

A sketch shows that the outer expansion looks like $f = \mathrm{e}^{-x}$ and the inner expansion like $F = 1 - \mathrm{e}^{-y}$. Using these two functions plotted on the same graph (using a particular value of ϵ such as 0.1). Each approximation is valid in its own region but they are both valid in an "overlap region" between the two.

We say that the expansions "match" in the sense than

$$\lim_{x \to 0} f_0(x) = 1 = \lim_{y \to \infty} F_0(y)$$

Loosely we interpret matching:, 'as we go into the boundary layer, the outer expansion tends to 1' and 'as we go out of the boundary layer, the inner expansion tends to 1 (the term e^{-y} becomes negligible)'

This loose statement can be made much more precise. One can think of such matching as follows: the inner and outer expansions should agree in a fuzzy 'intermediate' region such that $\mathrm{O}\,(\epsilon) \ll x \ll \mathrm{O}\,(1)$ - where $x \ll 1$ and $y \gg 1$.

Formally we can find asymptotic expansions in the inner and outer region and match them together. For some cases it is desirable to try to create a single function that is a reasonable approximation everywhere. Such a function is called a "composite expansion" and can be made in some circumstances, but not always. We can construct a *composite* expansion, by forming 'inner expansion + outer expansion - common limit' (the common limit should be subtracted to remove double-counting). This gives a uniform description

$$
\begin{aligned}
\text{composite expansion} \quad &= \quad f_0 + F_0 - 1 \\
&= \quad \underbrace{\mathrm{e}^{-x} + 1 - \mathrm{e}^{-x/\epsilon} - 1}_{\text{need to put both in terms of variable } x} \\
&= \quad \mathrm{e}^{-x} - \mathrm{e}^{-x/\epsilon}
\end{aligned}
$$

which is a good approximation to the exact solution

$$\frac{\mathrm{e}^{-x} - \mathrm{e}^{-x/\epsilon}}{1 - \epsilon} \ .$$

The method used to examine the ODE above is a specific example of a more general method of *matched asymptotic expansions*(MAEs).

7.5.3 Getting the expansion from the ODE

In the last example, we compared an approximate solution to one we found exactly, the whole point of MAEs is to use it on problems where we don't already know the answer (can't find it exactly, *etc.*).

In the previous example we found the outer expansion by assuming a very simple form for the asymptotic expansion (a Taylor series) of the ODE

$$\epsilon f' + f = e^{-x}$$

to find the solution

$$f \sim e^{-x} + \epsilon e^{-x} + O\left(\epsilon^2\right) \ .$$

Now the exact solution indicated that we should consider a narrow region near $x = 0$ and so we introduced a rescaling with $x = \epsilon y$ with $y = O(1)$. If we now take the ODE problem and do this rescaling we find that it becomes

$$\frac{\mathrm{d}F}{\mathrm{d}y} + F = e^{-\epsilon}y \sim 1 - \epsilon y + \cdots, \quad F(0) = 0$$

Now we can assume a simple Taylor expansion for the solution in this inner region by taking

$$F = F_0 + \epsilon F_1 + \cdots .$$

Putting the assumed solution in the ODE problem and equating powers of ϵ gives us

$$\frac{\mathrm{d}F_0}{\mathrm{d}y} + F_0 = 1, F_0(0) = 0 \Rightarrow F_0 = 1 - e^{-y}$$

and

$$\frac{\mathrm{d}F_1}{\mathrm{d}y} + F_1 = -y, F_1(0) = 0 \Rightarrow F_1 = 1 - y - e^{-y}$$

and so on for higher terms.

Note that we have been able to impose the boundary conditions on the problem near $x = 0$ whereas we could not impose them on the outer solution. It is important to note when a boundary layer might be present in a problem and hence need to be carefully accounted for. A common sign that a boundary layer (BL) is present is a power of ϵ multiplying the *highest derivative* in the ODE.

If, as $\epsilon > 0$ the order of ODE is reduced the problem is analogous to ϵ prefactor of the highest power, x^2, in the quadratic equation example we showed earlier. In general problems where the highest power or the highest derivative disappear when using a simple expansion are called *singular perturbation* problems

In the example earlier the only place there could be a BL is at $x = 0$ so that matching was 'automatic'. We found an inner solution and and outer solution but did not have to do anything to make them join smoothly.

There are two issues that need to be considered. Firstly, how to match the different solutions in the inner region and the outer region together when we are solving an ODE (and we do not know the solution and where matching is NOT automatic). Secondly we need some method to determine if there is a boundary layer and where it might be (it could be at either end of the domain, possible at both ends of the domain and in some very difficult problem in the interior of the region). Here we just consider cases where there is one boundary layer at one end of the region.

Example. Solve

$$\epsilon f'' + f' = 1, \quad 0 < x < 1 \quad (\epsilon > 0)$$
$$f(0) = f(1) = 0$$

Again you can find the exact solution:

$$f = x + A + B e^{-x/\epsilon}$$

where the boundary conditions give $A + B = 0$, $1 + A + B e^{-1/\epsilon} = 0$.

This indicates that there is a boundary layer near $x = 0$, but if we didn't have the exact solution, we wouldn't know this. However, let's find the solution knowing this and discuss why it is near $x = 0$ later.

Outer solution:

$$f \sim f_0 + \epsilon f_1 + \cdots \Rightarrow f_0' = 1 \Rightarrow f_0 = x + a$$

Because this is a first order ODE we can satisfy *one* of the two boundary conditions; let's satisfy the one at $x = 1$ (will get back to this soon)

$$1 + a = 0 \Rightarrow a = -1 \Rightarrow f_0 = x - 1$$

Then $\lim_{x \to 0^+} f_0 = -1 \neq 0$, so we don't satisfy the boundary condition at $x = 0$ and there must be a BL.

Now we find the *size* of the BL by scaling $x = \delta y$, where δ will depend on ϵ. Putting this change of independent variables into the problem gives

$$\frac{\epsilon}{\delta^2} \frac{\mathrm{d}^2 F}{\mathrm{d}y^2} + \frac{1}{\delta} \frac{\mathrm{d}F}{\mathrm{d}y} = 1$$

Now consider the *dominant balance*. In particular we want to choose δ to be very small and to make the highest derivative be in the dominant balance.

In this case this implies we take $\frac{\epsilon}{\delta^2} = \frac{1}{\delta}$, so $\delta = \epsilon$ so the ODE problem becomes

$$\frac{d^2 F}{dy^2} + \frac{dF}{dy} = \epsilon.$$

We can then assume a simple expansion: $F = F_0 + \epsilon F_1 + \cdots$ to give

$$F_0'' + F_0' = 0 \Rightarrow F_0 = c(1 - e^{-y}),$$

which satisfies $F_0(0) = 0$.

Matching

Having found the inner and the outer expansion we *match* these two parts of the solution together to make them smoothly join. The outer limit of F_0 is c (outer limit of inner solution) and it must equal the inner limit of f_0 which is -1 (the inner limit of outer solution).

Hence matching tells us that we must take $c = -1$. This then gives us

$$F_0 = e^{-y} - 1 .$$

If we wanted to we could now create a composite expansion

$$f_0 + F_0 - (-1) = x + e^{-x/\epsilon} - 1,$$

which is a good approximation of the exact solution.

7.5.4   Locating the position of possible Boundary Layers

The analysis of the ODE and the boundary layer that needed to be considered was guided by the exact solution. But what might be ways of finding where a boundary layer is needed??

OK, so why was the layer near $x = 0$ rather than $x = 1$? Basically, we want the inner solution to *decay* to a constant as we leave the BL and the sign in the inner region dominant balance ODE will ensure decay as x increases.

It is worth examining what would happen if we assumed the boundary layer was near $x = 1$. The first thing we would need to do is to rescale x to consider points near $x = 1$ and this could be done by taking $x = 1 + \delta(\epsilon)\xi$ where $\xi \leq 0$ is the new independent variable. A dominant balance argument for the highest derivative would then reveal $\delta(\epsilon) = \epsilon$ so that $x = 1 + \epsilon\xi$, $\xi < 0$. Assuming a simple expansion and equating powers of ϵ would then give us

$$\frac{d^2 F_0}{d\xi^2} + \frac{dF_0}{d\xi} = 0, \qquad F_0(0) = 0 \quad -\infty < \xi < 0 .$$

The solution to this problem is $F_0 = a(1 - \mathrm{e}^{-\xi})$ and this *grows* exponentially as ξ becomes large and negative. Hence it is not possible to "match" this solution to the solution in the outer region (except by taking $a = 0$ in which case there is no boundary layer to analyse).

Note. For the case $\epsilon < 0$, the BL *is* at $x = 1$.

7.6 Concluding remarks on asymptotic expansions of DEs

There is lots of other perturbation theory which you can do in C6.3a.
Overview: dynamical models in physical applied maths represent interaction between (physical) procedures and 'universal' (or at least reasonable) 'laws'; they also often incorporate experimental data. These models quite often take the form of ODEs and PDEs.
Once a model has been written down it is possible to use scaling and nondimensionalisation methods to reveal dimensionless parameters (or dimensionless groups) in the problem. By examining particular physical situations we may identify that some of these parameters may be very large or very small. In such circumstances it may be possible to use the asymptotic methods outlined here to generate solutions to problems where no exact solution is possible. Such approaches can be very useful and complement numeric methods.
Examples of problems where asymptotic methods can be used (**The remaining part of these notes is non-examinable material**)

Example. Convection of heat in a pipe
A circular pipe is held at a constant temperature $T = 1$ and there is a flow of fluid around the pipe. The fluid is at temperature $T = 0$ a long way from the pipe. The problem is considered to be in two dimensions. We are interested in how much heat leaves the pipe and goes into the fluid.
The governing equation for steady movement of the heat is

$$Pe\ \vec{u} \cdot \nabla T = \nabla^2 T$$

where u is a vector field representing the velocity of the fluid and Pe is a nondimensional parameter (the Peclet number) dependent on the speed of the fluid, the conductivity of heat in the fluid, and the radius of the pipe.
A common situation is that $Pe \gg 1$ due to the high speed of the fluid. Then we might rewrite the problem as

$$\vec{u} \cdot \nabla T = \epsilon \nabla^2 T, \quad \text{where } \epsilon = \frac{1}{Pe} \ll 1$$

Our previous discussion, indicates that because the highest derivative would disappear if we did a simple expansion with $\epsilon \to 0$ we must expect boundary layers. In fact we can find that there are *boundary layers near the pipe* (the fluid far from the pipe is at $T = 0$ so we need some conduction to get from $T = 1$ on the pipe to $T \to 0$ away from the pipe)

Boundary layers can also occur in other regions such as near *stagnation points* (points there flow vanishes — like behaviour near equilibrium points. A less common situations is where the flow is very slow and $Pe \ll 1$ but it is instructive to consider. Then the problem might be written as

$$\epsilon \vec{u} \cdot \nabla T = \nabla^2 T, \quad \text{where } \epsilon = Pe \ll 1$$

here we expect a regular expansion to work *but* in two-dimensions the leading-order solution $T_0 \propto \log r$ $(r = |\vec{x}|)$, which is unbounded at ∞; this implies that there is a "boundary layer at ∞". Such a boundary layer can be considered by rescaling $\vec{x} = \frac{1}{\epsilon}\vec{y}$ to get the problem

$$\vec{u} \cdot \nabla_{\vec{y}} T = \nabla_{\vec{y}}^2 T$$

(here $\nabla_{\vec{y}}$ is the gradient with respect to $\vec{y}$) with 'point source' behaviour at the origin. This is examined further in later courses.

In *quantum mechanics* you have the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial \tau} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(x, \tau)\Psi \ .$$

Semiclassics: what happens as $\hbar \to 0$? (i.e., use asymptotic analysis to try to recover classical mechanics descriptions from quantum ones!)

In general, whenever you are dealing with descriptions at multiple physical scales, asymptotics and perturbation theory provide much of the mathematical language to help address such issues!

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