# Part A: Differential Equations 2 HT2019

**Problem Sheet 1**

1. <u>Reduction of order and Variation of Parameters.</u>
   Define
   $$\mathfrak{L}y(x) \equiv x^2 y''(x) - x(x+2)y'(x) + (x+2)y(x), \quad 1 < x < 2.$$
   Check that $y(x) = x$ is a solution of $\mathfrak{L}y = 0$ and use reduction of order to find the general solution. Hence solve the following problem by variation of parameters:
   $$\mathfrak{L}y(x) = x^3, \qquad y(1) = 0, \quad y(2) = 0.$$

2. <u>Green's function via Variation of Parameters.</u>
   Use variation of parameters to solve the problem
   $$y''(x) - 2y'(x) + 2y(x) = f(x), \qquad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0, \qquad (\star)$$
   where $f$ is a given continuous function. Show that the solution can be written in the form
   $$y(x) = \int_0^{\pi/2} g(x,\xi) f(\xi)\,\mathrm{d}\xi$$
   and determine the Green's function $g$. Evaluate the integral when $f(x) = \mathrm{e}^x$ and check that the resulting expression for $y$ does indeed satisfy $(\star)$.

3. <u>Adjoint.</u>
   For each of the problems below, use the adjoint relation, $\langle \mathfrak{L}y, w\rangle \equiv \langle y, \mathfrak{L}^* w\rangle$, to determine the differential operator and boundary conditions for the adjoint problem. In each case state whether the operator and/or the full system is self-adjoint.

   (a) $\mathfrak{L}y = y''$, $\qquad 2y(0) + y'(0) = 0, \qquad y(1) + y'(1) = 0.$

   (b) $\mathfrak{L}y = y'''' - y'$, $\qquad y'(0) - y''(0) = 0, \qquad y'''(0) = 0, \qquad y(1) = 0, \qquad y'(1) - y'''(1) = 0.$

4. <u>Computing Green's function.</u> Obtain the Green's function for the following operators, using the delta function construction:

   (a) $\mathfrak{L}y = -y''$, $\quad 0 < x < 1, \quad y(0) - y'(1) = 0, \quad y(0) + y(1) = 0.$

   (b) $\mathfrak{L}y = y'' - y$, $\quad 0 < x < 2\pi, \quad y(0) - y(2\pi) = 0, \quad y'(0) - y'(2\pi) = 0.$

   In (b), what goes wrong if we change the operator to $\mathfrak{L}y = y'' + y$ (for the same boundary conditions)? Why?

5. Infinite domains. Obtain via the delta function the Green's function for each of the following BVPs, and use it to express the solution.

   (a) $\mathfrak{L}y(x) = y''(x) - \mu^2 y'(x) = f(x), \quad -\infty < x < 0,$

   where $\mu > 0$ is a real constant, with boundary conditions $y(0) = 1$ and $y \to 0$ as $x \to -\infty$.

   (b) $\mathfrak{L}u(x) \equiv u''(x) - \left(1 + x^2\right) u(x) = f(x), \quad -\infty < x < \infty,$

   with the boundary conditions $u(x) \to 0$ as $x \to \pm\infty$.

   [*Hint: Show that* $u(x) = \mathrm{e}^{x^2/2}$ *satisfies* $\mathfrak{L}u = 0$. *You may use without proof the fact that*

$$\int_{-\infty}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t = \sqrt{\pi}.]$$

6. FAT and Existence. Determine the parameter values $(A, B)$ that yield existence of a solution for each of the following inhomogeneous BVPs.

   (a) For $0 \le x \le 2\pi$:

$$y''(x) + y(x) = A\sin x + B\cos x + 2\sin\left(x + \frac{\pi}{3}\right) + \sin^3 x, \qquad y(0) = y(2\pi), \qquad y'(0) = y'(2\pi).$$

   (b) For $0 \le x \le 1$:

$$y''(x) + 2y'(x) + y(x) = 1, \qquad y'(0) + y(0) = A, \qquad y'(1) + y(1) = 3.$$

[*Hint: Note that the problem in* (a) *is fully self-adoint. In* (b), *show that the homogeneous adjoint problem has solution* $w(x) = \mathrm{e}^x$.]