UNIVERSITY OF OXFORD

B4.2 Functional Analysis II
Consultation Session 2

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Paper 2021–Q3(a)

Given

- X: a complex Hilbert space.
- $A: X \to X$ a bounded linear operator.
- $\mathrm{Im}\, A$ is closed.

Goal: Which of the following statements are true or false?

(i) 0 is not an element of $\sigma_c(A)$.

(ii) If $\lambda \in \sigma_{ap}(A) \setminus \sigma_p(A)$, then $\lambda \in \sigma_c(A)$.

(iii) If A is isometric, then $\sigma(A)$ is a subset of the closed unit disk.

(i) True. $0 \in \sigma_c(A)$ means A is injective and $\mathrm{Im}\, A$ is a proper dense subspace of X. This latter case can't hold as $\mathrm{Im}\, A$ is closed.

(ii) False. $\sigma_{ap}(A)$ may contain elements of $\sigma_r(A)$.

(iii) True. As $\|A\| = 1$, $\mathrm{rad}(\sigma(A)) \le 1$.

Paper 2021–Q3(b)

Given

- $X = L^2(-\pi, \pi)$.
- $A \colon X \to X$ a bounded linear operator.
- $\mathrm{Im}\, A$ is closed.
- Af is a trigonometric polynomial for every $f \in X$.

Goal: Show that $\mathrm{Im}\, A$ is finite-dimensional.

- Let $Y = \mathrm{Im}\, A$ and V_N be the space of trigonometric polynomials of degree at most N. Then $Y = \bigcup_N (Y \cap V_N)$.
- Since Y is closed, the Baire category theorem gives an N so that $\overline{Y \cap V_N}$ has non-empty interior in Y.
- As Y and V_N are closed, this implies that $Y \cap V_N$ has non-empty interior in Y. Since $Y \cap V_N$ is a subspace of Y, this implies that $Y \cap V_N = Y$ (why?), i.e. $Y \subset V_N$. Hence $\dim Y < \infty$.

Paper 2021–Q3(c)

Given

- $X = L^2(-\pi, \pi)$.
- $A \colon X \to X$ a bounded linear operator.
- $\mathrm{Im}\, A$ is closed.
- Af is a trigonometric polynomial for every $f \in X$.

Goal: Show

- $0 \in \sigma_p(A)$.
- $\sigma_{ap}(A) \subset \sigma_p(A)$.
- $\sigma_p(A)$ contains at most $d + 1$ eigenvalues, where $d = \dim Y$.

- Let W be any finite dimensional subspace of X containing $Y = \mathrm{Im}\, A$ as a proper subspace, and let $B = A|_W$ so that B maps W into $Y \subset W$. Since $\dim Y < \dim W$, we have by the rank-nullity theorem that B is not injective: there exists $w \in W$ such that $0 = Bw = Aw$. This shows that $0 \in \sigma_p(A)$.

Paper 2021–Q3(c)

- Pick $\lambda \in \sigma_{ap}(A)$ and we need to show that $\lambda \in \sigma_p(A)$. If $\lambda = 0$, we are done from the above. We assume henceforth that $\lambda \neq 0$.
- By definition, there exists $(x_n) \subset X$ such that $\|x_n\| = 1$ and $\lambda x_n - Ax_n \to 0$.
- Note that $(Ax_n) \subset Y$ is a bounded sequence in a finite dimensional vector space. Therefore, passing to a subsequence if necessary, we may assume that (Ax_n) is convergent.
- As $\lambda x_n - Ax_n \to 0$, we have that (λx_n) and so (x_n) are strongly convergent (since $\lambda \neq 0$).
- The strong limit x of (x_n) satisfies $\|x\| = 1$ and $\lambda x - Ax = 0$. We conclude that $\lambda \in \sigma_p(A)$.

Paper 2021–Q3(c)

- Next, note that every eigenvector of A corresponding to a non-zero eigenvalue belongs to Y.
- Since eigenvectors corresponding to different eigenvalues are linearly independent, the number of nontrivial eigenvalues must be no larger than the dimension d of Y.
- So $\sigma_p(A)$ contains no more than $d + 1$ elements.

Paper 2021–Q3(d)

Given

- $X = L^2(-\pi, \pi)$.
- $A\colon X \to X$ a bounded linear operator.
- $\mathrm{Im}\, A$ is closed.
- Af is a trigonometric polynomial for every $f \in X$.
- $\sigma(A)$ contains exactly two elements.

Goal:

- Show that if A is self-adjoint, then there is a constant $c \in \mathbb{R} \setminus \{0\}$ such that $(1/c)A$ is the orthogonal projection operator onto its range.
- If A is not self-adjoint, must A be a multiple of a (non-necessarily orthogonal) projection operator, i.e. must there be a $c \in \mathbb{C} \setminus \{0\}$ such that $A^2 = cA$? Give either a proof or a counter-example.

Paper 2021–Q3(d)

- If A is self-adjoint, then $\sigma_r(A)$ is empty and $\sigma(A) = \sigma_{ap}(A)$ is real. Together with (c), we have that all elements of $\sigma(A)$ are real eigenvalues of A and one of which is 0. Since $\sigma(A)$ contains exactly 2 elements, $\sigma(A) = \{0, \lambda\}$ with $\lambda \in \mathbb{R} \setminus \{0\}$.
- Let $Z = Y^\perp$. Then $\mathrm{Ker}\, A = \mathrm{Im}\, A^\perp = Z$. It follows that $\hat{A} := A|_Y : Y \to Y$ is a bijection.
- Note that $\hat{A}$ is self-adjoint on Y:

$$\langle \hat{A}x, y\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle = \langle x, \hat{A}y\rangle \text{ for all } x, y \in X.$$

  As Y is finite dimensional, Y has a basis consisting of eigenvectors of $\hat{A}$, which are also eigenvectors of A.
- From the above, $\hat{A} = \lambda I|_Y$ and so A is a multiple of the orthogonal projection into Y.

Paper 2021–Q3(d)

- If A isn't self-adjoint, it is not necessary that A is a multiple of a projection operator.
- For example, consider the operator

$$A\Big(\sum_n c_n e^{inx}\Big) = 2c_0 + c_1 e^{-ix}.$$

It is clear that $Y = \mathrm{Span}(1, e^{-ix})$, hence closed.

- A computation gives

$$A^2\Big(\sum_n c_n e^{inx}\Big) = 4c_0 \text{ and } A^3\Big(\sum_n c_n e^{inx}\Big) = 8c_0.$$

In particular, A satisfies $A^3 - 2A^2 = 0$, which implies $\sigma(A) \subset \{0, 2\}$.

- Clearly $A(1) = 2$ and $A(e^{2ix}) = 0$, so $\sigma(A) = \{0, 2\}$. But certainly A^2 is not a multiple of A.

Paper 2015–Q2

Let X be a Hilbert space.

(a) Prove that a subset E of X is norm-bounded if for each $x \in X$ there exists a constant M_x such that

$$|\langle x, y\rangle| \le M_x \text{ for all } y \in E.$$

(b) Let $T : X \to X$ be a bijective continuous linear operator. Prove that there is a constant $m > 0$ such that $y \in X$ and $\|T^* y\| = 1$ together imply $\|y\| \le m$. Hence prove that T has a continuous inverse.

(c) Deduce that a surjective bounded linear operator from X to X maps open sets to open sets.

Paper 2015–Q2(a)

Let X be a Hilbert space. Prove that a subset E of X is norm-bounded if for each $x \in X$ there exists a constant M_x such that

$$|\langle x, y\rangle| \leq M_x \text{ for all } y \in E. \qquad (*)$$

- For $y \in X$, define a linear functional $\ell_y \in X^*$ by $\ell_y(x) = \langle x, y\rangle$. Note that $\|\ell_y\| = \|y\|$ in view of the Cauchy-Schwarz inequality (why?).
- Let $\mathscr{F} = \{\ell_y : y \in E\}$. Clearly E is bounded iff $\mathscr{F}$ is bounded in X^*.
  On the other hand, by the principle of uniform boundedness, $\mathscr{F}$ is bounded in X^* iff (*) holds. The conclusion follows.

Paper 2015–Q2(b)

Let X be a Hilbert space and $T : X \to X$ be a bijective continuous linear operator. Prove that there is a constant $m > 0$ such that $y \in X$ and $\|T^* y\| = 1$ together imply $\|y\| \leq m$. Hence prove that T has a continuous inverse.

- Let $E = \{y : \|T^* y\| = 1\}$. We need to show that E is bounded.
- By (a), we need to show that for every $x \in X$, the set $\{\langle x, y\rangle : y \in E\}$ is bounded.
- Fix $x \in X$. To make $T^* y$ shows up, we write $x = Tz$ which is possible as T is bijective. Then

$$|\langle x, y\rangle| = |\langle Tz, y\rangle| = |\langle z, T^* y\rangle$$

which by the Cauchy-Schwarz inequality is bounded from above by

$$\leq \|z\|\|T^* y\| = \|z\|.$$

We conclude from (a) that E is bounded.

Paper 2015–Q2(b)

Let X be a Hilbert space and $T : X \to X$ be a bijective continuous linear operator. Prove that there is a constant $m > 0$ such that $y \in X$ and $\|T^*y\| = 1$ together imply $\|y\| \leq m$. Hence prove that T has a continuous inverse.

- From the above, we have that, for every $y \in X$, $\|T^*(y/\|T^*y\|)\| = 1$ and so $\|y\|/\|T^*y\| \leq m$ i.e.

$$\|T^*y\| \geq m^{-1}\|y\| \text{ for all } y \in X. \qquad (**)$$

- We knew that this implies $\mathrm{Im}\, T^*$ is closed. Since $(\mathrm{Im}\, T^*)^\perp = \mathrm{Ker}\, T = 0$, we have that $\mathrm{Im}\, T^* = X$. Also $\mathrm{Ker}\, T^* = (\mathrm{Im}\, T)^\perp = 0$. So T^* is invertible with bounded inverse (in view of (**)).
- Properties of adjoints imply that T has bounded inverse.

Paper 2015–Q2(c)

Let X be a Hilbert space and $T : X \to X$ be a surjective continuous linear operator. Prove that T maps open sets to open sets.

- We know (why?) that T is open if there exists $\delta_0 > 0$ such that

$$T(B(0,1)) \supseteq B(0,\delta_0).$$

- Let $Y = \mathrm{Ker}\, T$ and $Z = Y^\perp$. Both of these are Hilbert subspaces of X.
- Let $S = T|_Z : Z \to X$ which is a bijective bounded linear operator. Though the domain and target spaces are different, the same proof of (b) gives that S has a bounded inverse.
- This means that there exists $\delta > 0$ so that $\|Tz\| = \|Sz\| \geq \delta \|z\|$ for all $z \in Z$.
- Now if $\|x\| < \delta$ and $x = Tz$, then $\|z\| \leq \delta^{-1}\|x\| = 1$, i.e. $B(0,\delta) \subseteq T(B(0,1))$.

Paper 2008 – Q8(a)

Given

- V: a complex Hilbert space.
- $T \in \mathscr{B}(V)$ is self-adjoint.
- $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Goal

- Prove that $\|\lambda v - Tv\| = \|\bar{\lambda} v - T^* v\| \geq |\mathrm{Im}\lambda| \|v\|$. $(\star)$
  Deduce that $\lambda I - T$ and $(\lambda I - T)^*$ are injective.
- Prove that $\mathrm{Im}\,(\lambda I - T)$ is closed in V, and by considering the orthogonal complement $[(\lambda I - T)(V)]^\perp$, show that $\lambda I - T$ is surjective.
- Show that $\lambda I - T$ has a bounded inverse $(\lambda I - T)^{-1}$ with $\|(\lambda I - T)^{-1}\| \leq |\mathrm{Im}\lambda|^{-1}$. Deduce that $\sigma(T) \subset \mathbb{R}$.

Paper 2008 – Q8(a)

- Suppose for the moment that $(\star)$ has been shown.
  It is clear that $(\star)$ implies that $\lambda I - T$ and $(\lambda I - T)^*$ are injective.
  Furthermore, this coercivity implies that the range $\mathrm{Im}\,(\lambda I - T)$ is closed (see Session 1).
  Since $[(\lambda I - T)(V)]^\perp = \mathrm{Ker}(\bar{\lambda} I - T^*) = 0$, we thus have that $\lambda I - T$ is surjective and hence bijective.
  Recalling $(\star)$ again, we have $\|(\lambda I - T)^{-1}\| \leq |\mathrm{Im}\lambda|^{-1}$, and so $\lambda \notin \sigma(T)$.
  Since $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is arbitrary, we have $\sigma(T) \subset \mathbb{R}$.
  So the main issue is to show $(\star)$.

Paper 2008 – Q8(a)

- Switching the role of λ and $\bar{\lambda}$, it is enough to estimate $\|\lambda v - Tv\|$.
- Write $\lambda = a + ib$. We compute

$$\begin{aligned}\|\lambda v - Tv\|^2 &= \|av - Tv + ibv\|^2 \\ &= \|av - Tv\|^2 + b^2\|v\|^2 \\ &\qquad \underbrace{-ib\langle av - Tv, v\rangle + ib\langle v, av - Tv\rangle}_{=0 \text{ as } aI - T \text{ is self-adjoint}} \\ &= \|av - Tv\|^2 + b^2\|v\|^2 \geq |Im\lambda|^2\|v\|^2,\end{aligned}$$

which proves $(\star)$.

- A related (but easier) inequality: If U is isometric, then $\|\lambda v - Uv\| \geq \big||\lambda| - 1\big|\|v\|$.

Paper 2014–Q2(c)

Let X be a Banach space.

(i) Bookwork: Boundedness of projection map.

(ii) For $n \geq 1$, let Y_n and Z_n be closed subspaces of X such that

$$Y_n \subseteq Y_{n+1}, \quad Z_n \supseteq Z_{n+1}, \quad X = Y_n \oplus Z_n.$$

Let P_n be given by $P_n(y_n + z_n) = y_n$ for $y_n \in Y_n$ and $z_n \in Z_N$. Assume that for each $x \in X$, the limit $\lim_{n\to\infty} P_n x$ exists and denote this limit by Px. Prove that P is a bounded projection and that

$$\mathrm{Im}\, P = \overline{\bigcup_{n\geq 1} Y_n} \quad \text{and} \quad \mathrm{Ker}\, P = \bigcap_{n\geq 1} Z_n.$$

Paper 2014–Q2(c)

- It is straightforward to check that P is linear.
- For each x, $\{P_n x\}$ is bounded (since it's convergent), we have by the principle of uniform boundedness that $\{P_n\}$ is bounded in $\mathscr{B}(X)$, i.e. there exists $M \geq 0$ such that $\|P_n\| \leq M$ for all n.
- Using $P_n^2 = P_n$ and

$$\|P_n(P_n x - Px)\| \leq M\|P_n x - Px\| \to 0,$$

we have $P^2 = P$.
- Next,

$$\|Px\| \leq \underbrace{\|(P_n x - Px)\|}_{\to 0} + \underbrace{\|P_n x\|}_{\leq M\|x\|} \quad \text{and so} \quad \|Px\| \leq M\|x\|.$$

This means P is bounded.
- Now let $Y = \overline{\cup Y_n}$ and $Z = \cap Z_n$ which are closed subspaces of X. We need to show $\mathrm{Im}\, P = Y$ and $\mathrm{Ker}\, P = Z$.

Paper 2014–Q2(c)

- Let us show that $\operatorname{Ker} P = Z$. For each $z \in Z$, we have $z \in Z_n$ for all n and so $P_n z = 0$ for all n. In particular, $Pz = \lim P_n z = 0$.
  Conversely, suppose $Pz = 0$. Then $z = \lim(I - P_n)z$. Note that $(I - P_n)z \in Z_n \subseteq Z_m$ if $n \geq m$. It follows that the sequence $((I - P_n)z)$ eventually belongs to Z_m for each m. Since Z_m are closed, we have that $z = \lim(I - P_n)z$ belongs to all Z_m, i.e. $z \in Z$.
- Next, we show $\operatorname{Im} P = Y$.
  If $y = Px = \lim P_n x$, then since $(P_n x) \subset \cup Y_n$ we have $y \in Y$. So $\operatorname{Im} P \subseteq Y$.
  Take $y \in \cup Y_n$ so that $y \in Y_m$ for some m. Then $y \in Y_n$ for all $n \geq m$. It follows that $P_n y = y$ for $n \geq m$ and so $Py = y$. By continuity, we have $Py = y$ for all $y \in Y$ and so $Y \subseteq \operatorname{Im} P$.

Paper 2017–Q2

(a) (ii) Let $X = C([0,1])$. Define $A_n : X \to X$ so that $(A_n x)(t) = x(t^{1+\frac{1}{n}})$. Show that A_n converges strongly to the identity operator on X (i.e. $A_n x \to x$ for all $x \in X$.)

(b) (i) Let X be a Banach space, Y be a normed space, $T \in \mathscr{B}(X,Y)$. Assume $\overline{T(B_X(0,1))} \supseteq B_Y(0,\varepsilon)$ for some $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that $T(B_X(0,1)) \supseteq B_Y(0,\delta)$.

(iii) Let X and Y be Banach spaces and $T \in \mathscr{B}(X,Y)$. Prove that if $T(X)$ is not closed in Y, then $T(X)$ is a countable union of nowhere dense subsets of Y.

(iv) Show that $C([0,1])$ is a countable union of nowhere dense subsets of $L^2(0,1)$.

Paper 2017–Q2(a)(ii)

Let $X = C([0,1])$. Define $A_n : X \to X$ so that $(A_n x)(t) = x(t^{1+\frac{1}{n}})$. Show that A_n converges strongly to the identity operator on X (i.e. $A_n x \to x$ for all $x \in X$.)

- Fix $x \in X$. We need to show

$$\sup_{t\in[0,1]} |x(t^{1+\frac{1}{n}}) - x(t)| \to 0 \text{ as } n \to \infty.$$

- By uniform continuity of x, it suffices to show

$$\sup_{t\in[0,1]} |t^{1+\frac{1}{n}} - t| \to 0 \text{ as } n \to \infty.$$

- Fix some small $\epsilon > 0$.
  + If $t \leq \epsilon$, then $|t^{1+\frac{1}{n}} - t| < \epsilon$ for all n.
  + If $t > \epsilon$, then $|t^{1+\frac{1}{n}} - t| \leq |t^{\frac{1}{n}} - 1| \leq 1 - \epsilon^{\frac{1}{n}} < \epsilon$ for all large n.

Paper 2017–Q2(b)(i)

Let X be a Banach space, Y be a normed space, $T \in \mathscr{B}(X, Y)$. Assume $\overline{T(B_X(0,1))} \supseteq B_Y(0,\varepsilon)$ for some $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that $T(B_X(0,1)) \supseteq B_Y(0,\delta)$.

- We prove the statement with $\delta = \varepsilon/2$.
- As $\overline{T(B_X(0,1))} \supset B_Y(0,\varepsilon)$, we have $\overline{T(B_X(0,r))} \supset B_Y(0,\varepsilon r)$.
- Take $y \in B_Y(0,\varepsilon/2) \subset \overline{T(B_X(0,1/2))}$.
- Take $x_1 \in B_X(0,1/2)$ such that $\|y - Tx_1\| < \varepsilon/4$. Then

$$y - Tx_1 \in B_Y(0,\varepsilon/4) \subset \overline{T(B_X(0,1/4))}.$$

- Take $x_2 \in B_X(0,1/4)$ such that $\|(y - Tx_1) - Tx_2\| < \varepsilon/8$.

Paper 2017–Q2(b)(i)

- Inductively, we obtain $x_k \in B_X(0, 2^{-k})$ such that

$$\|y - T(x_1 + \ldots + x_k)\| < \varepsilon\, 2^{-k-1}.$$

- Easy to check: The series $\sum x_k$ converges to some s satisfying $y = Ts$ and

$$\|s\| < \sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} 2^{-k} = 1, \text{ i.e. } s \in B_X(0,1).$$

We have thus shown that $B_Y(0, \varepsilon/2) \subset T(B_X(0,1))$.

Paper 2017–Q2(b)(iii)

Let X and Y be Banach spaces and $T \in \mathscr{B}(X, Y)$. Prove that if $T(X)$ is not closed in Y, then $T(X)$ is a countable union of nowhere dense subsets of Y.

- As $TX = \cup_n T(B_X(0, n))$, it suffices to show that $T(B_X(0, 1))$ is nowhere dense.
- Suppose by contradiction that $\overline{T(B_X(0, 1))}$ has non-empty interior, i.e. $\overline{T(B_X(0, 1))} \supset B_Y(y_0, r_0)$ for some $r_0 > 0$.
- Then we also have that $\overline{T(B(0, 1))} \supset B_Y(-y_0, r_0)$, which in turns implies that

$$\overline{T(B_X(0, 1))} \supset B_Y(0, r_0) = \frac{1}{2}(B_Y(y_0, r_0) + B_Y(-y_0, r_0)).$$

- By (i), we then have $T(B_X(0, 1)) \supset B_Y(0, \delta)$ for some $\delta > 0$.
- This implies that $T(B_X(0, n)) \supset B_Y(0, \delta n)$ and so $TX = Y$, contradicting the fact that TX is not closed.

Paper 2017–Q2(b)(iv)

Show that $C([0,1])$ is a countable union of nowhere dense subsets of $L^2(0,1)$.

- Let $X = C[0,1]$ and $Y = L^2(0,1)$ and equip them with their standard norms to make them Banach spaces.
- Let $T : X \to Y$ be the natural injection, which is clearly bounded linear.
- Clearly T is not surjective and so, by (iii), $X = TX$ is a countable union of nowhere dense sets in $L^2(0,1)$.