

PART A TOPOLOGY
HT 2019
EXERCISE SHEET 1

For tutorials in week 3

Exercise 1. Let $\mathcal{T}_{\text{left}}$ be the family of subsets U of \mathbb{R} with the property that for every $x \in U$ there exists $\varepsilon > 0$ such that $(x - \varepsilon, x] \subseteq U$. Prove that $\mathcal{T}_{\text{left}}$ is a topology on \mathbb{R} .

What is the closure of $(0, 1)$ with respect to this topology?

Closure, interior, accumulation points

Exercise 2. Let (X, d) be a metric space and A a non-empty subset of X . For every $x \in X$, define the distance from x to A , $\text{dist}(x, A)$, by

$$\text{dist}(x, A) = \inf_{a \in A} d(x, a).$$

Prove that $x \in \bar{A}$ if and only if $\text{dist}(x, A) = 0$.

Exercise 3. Let (X, d) be a metric space.

- (1) Prove that a closed ball is a closed set in (X, \mathcal{T}_d) , where \mathcal{T}_d is the topology defined by the metric d . Prove that the closure of an open ball is contained in the closed ball.
- (2) Draw the picture of the closed ball with centre $(\frac{1}{2}, 0)$ and of radius $\frac{1}{2}$ in $(\mathbb{R}^2, \|\cdot\|_\infty)$.
- (3) Consider the subset $E = ([0, 1] \times \{0\}) \cup (\{0\} \times [0, 1])$ of \mathbb{R}^2 endowed with the metric induced from $(\mathbb{R}^2, \|\cdot\|_\infty)$. What are the open and the closed ball of centre $(\frac{1}{2}, 0)$ and of radius $\frac{1}{2}$ in this new metric space? Show that the closure of the open ball is a proper subset of the closed ball.
- (4) Let $(V, \|\cdot\|)$ be a normed real vector space. Show that in this case the closure of any open ball is a closed ball.

Exercise 4. Let $f: X \rightarrow Y$ be a continuous map between topological spaces, and let A and B be subsets of X such that $\bar{A} = \bar{B}$. Prove that $\overline{f(A)} = \overline{f(B)}$.

Exercise 5. Give examples of subsets A_1, A_2 and A_3 of \mathbb{R}^2 (with its standard topology) such that

- (1) $A'_1 = \mathbb{R}^2$ and $(\mathbb{R}^2 \setminus A_1)' = \mathbb{R}^2$;
- (2) $A'_2 \not\subseteq A_2$ and $A_2 \not\subseteq A'_2$;
- (3) $A'''_3 \neq A''_3 \neq A'_3 \neq A_3$.

Hausdorff spaces

Exercise 6. Let (X, \mathcal{T}) be a *Hausdorff* topological space and let A be a non-empty subset of X .

- (1) Prove that an open set $U \in \mathcal{T}$ has non-empty intersection with A if and only if U has non-empty intersection with \overline{A} .
- (2) Prove that a point x is an accumulation point of A (i.e. $x \in A'$) if and only if x is in $\overline{A \setminus \{x\}}$.
- (3) Prove that A' is closed in X .
- (4) Prove that $(\overline{A})' = A'$. Deduce that $(A')' \subseteq A'$.