Definitions of measurable sets, and regularity of Lebesgue measure

Three equivalent definitions

As mentioned in the course, there are numerous ways of defining Lebesgue measurable subsets of $\mathbb{R}$. The definition given in the lectures, and also in Zhongmin Qian's lecture notes, and in the book of Capinski & Kopp, is that a subset E of $\mathbb{R}$ is Lebesgue measurable if

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

for all subsets A of $\mathbb{R}$. We write $E \in \mathcal{M}_{\text{Leb}}$ when this holds.

An alternative definition, given in the book of Stein & Shakarchi and elsewhere, is that E is measurable if, for every $\varepsilon > 0$, there exists an open set U such that $E \subseteq U$ and $m^*(U \setminus E) < \varepsilon$. We write $E \in \mathcal{M}_{\text{SS}}$ when this holds. This definition can, in principle, be checked by identifying sets U for countably many $\varepsilon > 0$.

There is a third definition in Garling's book. He defines outer measure (denoted by λ^* in his case) in essentially the same way as in our course, but he also defines the *size* of a compact subset K of $\mathbb{R}$ to be $s(K) = m^*(U) - m^*(U \setminus K)$ where U is any bounded open subset of $\mathbb{R}$ containing K. Garling shows that $s(K)$ is independent of the choice of U and then defines the inner measure of $E \subseteq \mathbb{R}$ to be

$$m_*(E) = \sup \{s(K) : K \text{ compact}, K \subseteq E\}.$$

Then he defines E to be measurable if $m^*(E_n) = m_*(E_n)$ for all $n \geq 1$, where $E_n = E \cap [-n, n]$. We write $E \in \mathcal{M}_{\text{Gar}}$ when this holds. It is sufficient that $m^*(E) = m_*(E)$ if the common value is finite, but not if it is infinite.

We now show that these three definitions are equivalent, using facts from our course. We may use the notation $m(E)$ instead of $m^*(E)$ when $E \in \mathcal{M}_{\text{Leb}}$. For example, it is clear from our course that $s(K) = m(U) - (m(U) - m(K)) = m(K)$ for any compact set K and bounded open $U \supseteq K$.

First assume that $E \in \mathcal{M}_{\text{Leb}}$. By definition of m^* using open intervals, there exist open U_n such that $E_n \subseteq U_n$ and $m^*(U_n) < m^*(E_n) + \varepsilon 2^{-n}$. Let $U = \bigcup_n U_n$. Then U is open, $E \subseteq U$ and $U \setminus E \subseteq \bigcup_n (U_n \setminus E_n)$. Hence

$$m^*(U \setminus E) \leq \sum_n m^*(U_n \setminus E_n) = \sum_n (m(U_n) - m(E_n)) < \varepsilon.$$

So $E \in \mathcal{M}_{\text{SS}}$.

On the other hand, assume that $E \in \mathcal{M}_{\text{SS}}$. Then for each n, there exists an open set U_n such that $E \subseteq U_n$ and $m^*(U_n \setminus E) < n^{-1}$. Let $G = \bigcap U_n$, so G is (Borel) measurable, $E \subseteq G$ and $m^*(G \setminus E) = 0$. This implies that G and $G \setminus E$ are Lebesgue measurable, and hence so is $(\mathbb{R} \setminus G) \cup (G \setminus E) = \mathbb{R} \setminus E$. So $E \in \mathcal{M}_{\text{Leb}}$.

Now suppose that $E \in \mathcal{M}_{\text{Leb}} = \mathcal{M}_{\text{SS}}$. Let $n \geq 1$ and $\varepsilon > 0$. Since $E_n \in \mathcal{M}_{\text{SS}}$ there exists an open set U such that $E_n \subseteq U$ and $m^*(U \setminus E_n) < \varepsilon$, so $m(U) < m(E_n) + \varepsilon$. Replacing E_n by $[-n, n] \setminus E_n$, there exists an open set V such that $[-n, n] \setminus E_n \subseteq V$ and $m(V) < m([-n, n] \setminus E_n) + \varepsilon$, so $m(E_n \cap V) < \varepsilon$. Let $K = [-n, n] \cap (\mathbb{R} \setminus V)$, so K is compact, $E_n = K \cup (E_n \cap V)$, $K \subseteq E_n$ and $m(E_n) < m(K) + \varepsilon$. It follows that $m^*(E_n) = m(E_n) = m_*(E_n)$.

If $E \in \mathcal{M}_{\text{Gar}}$, then, for each n, there exist compact K_n and open U_n such that $K_n \subseteq E_n \subseteq U_n$ and $m(U_n \setminus K_n) = m(U_n) - m(K_n) < 2^{-n}$. Let

$$A = \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} K_n, \qquad B = \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} U_n.$$

Then A and B are (Borel) measurable, $A \subseteq E \subseteq B$ and $m(B \setminus A) = 0$ since $B \setminus A \subseteq \bigcup_{n \geq m}(U_n \setminus K_n)$ for any m. Thus E is the union of a Borel set A and a null set $E \setminus A$, so $E \in \mathcal{M}_{\text{Leb}}$.

Regularity of Lebesgue measure

As a consequence of the arguments above, the following hold for any Lebesgue measurable set E:

$$\begin{aligned} m(E) &= \inf\{m(U) : U \text{ open}, E \subseteq U\} && \text{(outer regularity)}, \\ &= \sup\{m(K) : K \text{ compact}, K \subseteq E\} && \text{(inner regularity)}. \end{aligned}$$

Outer regularity is almost immediate from the definition of m^*. Inner regularity follows from $m(E) = \sup_n m(E_n) = \sup_n m_*(E_n)$.

Example 2.4

For the set A described in Example 2.4 of the course, $m_*(A) = 0$. It is impossible to say what is the value of $m^*(A)$ because A is not specified explicitly. All we can say is that $0 < m^*(A) \leq 1$.

Priestley definition

Priestley defines a set $E \subseteq \mathbb{R}$ to be measurable if χ_E is the limit a.e. of a sequence of step functions ψ_n. This is equivalent to our definition by the (unproven) Theorem 3.10 in the lecture notes.

Etheridge definition

The property (0.1) is a convenient definition for deducing properties of measurable sets, but it is impossible to check it directly for individual sets E, because it involves all subsets of $\mathbb{R}$, including any which might be non-measurable. When Alison Etheridge gave the course in 2008, 2010 and 2011, she defined measurable sets by requiring (0.1) to hold not for arbitrary subsets A, but only for bounded intervals. She wrote that the two definitions are equivalent, but *that is not a very easy thing to show.*