RINGS AND MODULES HT19 – SHEET ONE

Rings. Ideals and Quotient Rings. Isomorphism Theorems. Chinese Remainder Theorem.

1.(i) Show that $2\mathbb{Z}$ and $3\mathbb{Z}$ are isomorphic groups but that the *rings* $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.
(ii) How many elements of $\mathbb{Z}_{144}$ are units?
(iii) What is the subring of $C(\mathbb{R})$ generated by x?

2. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbb{H} = \left\{ \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} : x, y \in \mathbb{C} \right\}.$$

(i) Show that $\mathbb{R}[A]$ is a subring of $M_2(\mathbb{R})$ which is isomorphic to $\mathbb{C}$.
(ii) Find complex matrices B and C such that $\mathbb{H} = \mathbb{R}[B, C]$. Show that $\mathbb{H}$ is non-commutative subring and that every non-zero element of $\mathbb{H}$ has an inverse in $\mathbb{H}$.

3. You are given that the following are CRIs. In each case, identify the identity 1_R, the units and any zero-divisors.
(i) R is the set of $n \times n$ diagonal complex matrices, with matrix addition and multiplication.
(ii) $R = \mathbb{Z}^2 = \{(m, n) : m, n \in \mathbb{Z}\}$ with addition and multiplication defined coordinate-wise.
(iii) $R = \mathbb{R}^{\mathbb{R}}$ is the set of functions from $\mathbb{R}$ to $\mathbb{R}$, with pointwise addition and multiplication.
(iv) [Harder] $R = C(\mathbb{R})$ is the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, with the usual pointwise addition and multiplication.

4. Which of the following I_k are ideals of $\mathbb{R}[x]$? If I_k is an ideal, find a monic polynomial p_k such that $I_k = \langle p_k \rangle$.
(i) $I_1 = \{a(x) \in \mathbb{R}[x] : a(2) = 0\}$.
(ii) $I_2 = \{a(x) \in \mathbb{R}[x] : a'(2) = 0\}$.
(iii) $I_3 = \{a(x) \in \mathbb{R}[x] : a(2) = a(3) = 0\}$.
(iv) $I_4 = \{a(x) \in \mathbb{R}[x] : a(1) = a'(1) = 0\}$.

5. Let A be a square matrix with entries in the field F, and let $m(x)$ denote the minimal polynomial of A. Show that

$$F[A] \cong \frac{F[x]}{\langle m(x) \rangle}.$$

Show that if A is a 2×2 real square matrix then $\mathbb{R}[A]$ is isomorphic to one of

$$\mathbb{R}, \qquad \mathbb{R} \times \mathbb{R}, \qquad \mathbb{R}[x]/\langle x^2 \rangle, \qquad \mathbb{C}.$$

6.(i) Write down the addition and multiplication tables for $\mathbb{Z}_2[x]/\langle x^2 + x + 1 \rangle$.

(ii) Let $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$. Show that the map

$$\phi : \mathbb{Z}[i] \to \mathbb{Z}_5 \text{ given by } a + bi \mapsto a + 2b \bmod 5$$

is a ring homomorphism. Show further that the kernel of ϕ is $\langle 2 - i \rangle$ and that $\mathbb{Z}[i]/\langle 2 - i \rangle \cong \mathbb{Z}_5$.

(iii) Find a zero-divisor in $\mathbb{Z}[i]/\langle 2 \rangle$.

7. Let

$$R_1 = \frac{\mathbb{Q}[x]}{\langle x^2 - 1 \rangle}, \qquad R_2 = \frac{\mathbb{Q}[x]}{\langle x^2 - 2 \rangle}.$$

Show that the map $\phi : R_1 \to \mathbb{Q}^2$ given by $p(x) \mapsto (p(1), p(-1))$ is an isomorphism. Deduce that R_1 is not an integral domain and find all the zero-divisors in R_1. How many roots does the equation $z^2 - 1 = 0$ have in R_1?

Show that R_2 is isomorphic to $\mathbb{Q}[\sqrt{2}]$ and hence that R_2 is a field.

8. Let $\phi : \mathbb{Z}[x] \to \mathbb{Q}$ be a non-zero ring homomorphism. Prove that $\phi(1) = 1$.

Prove also that, if $\phi(x) = u/v$ (where $v \in \mathbb{N} - \{0\}$ and $u \in \mathbb{Z}$), then a *necessary* condition for a rational number m/n (in lowest terms) to lie in the image of ϕ is that every prime divisor p of n be a divisor of v.

Deduce, or prove otherwise, that there are no ideals A in $\mathbb{Z}[x]$ such that $\mathbb{Z}[x]/A \cong \mathbb{Q}$.

Are there ideals A of $\mathbb{Z}[x]$ such that (i) $\mathbb{Z}[x]/A \cong \mathbb{Z}$? (ii) $\mathbb{Z}[x]/A \cong \mathbb{Z}_2$? (iii) $\mathbb{Z}[x]/A \cong \mathbb{Z}[i]$? (iv) $\mathbb{Z}[x]/A \cong \mathbb{Z}[\frac{1}{2}]$?

9. (Optional) (i) Use the Chinese remainder theorem to find all solutions $x \in \mathbb{Z}$ to the simultaneous congruences $x \equiv 8 \bmod 17$ and $x \equiv 3 \bmod 9$.

(ii) Find all solutions (if any) to the simultaneous congruences $x \equiv 13 \bmod 15$, $x \equiv 10 \bmod 21$, $x \equiv 3 \bmod 35$.

(iii) Find all solutions $x \in \mathbb{Z}_{143}$ to the quadratic $x^2 + 4x + 3 \equiv 0 \bmod 143$.