RINGS AND MODULES HT19 – SHEET FOUR

Smith normal form. Structure of finitely-generated modules Matrix canonical forms. Finite Abelian groups.

1. Which of the following facts (which are true of vector spaces) are true in general of modules over an ED? Justify your answers.
(i) In a free module, a linearly independent set can be extended to a basis.
(ii) In a free module, a minimal spanning set is a basis.
(iii) A square matrix is invertible (over the ring) if and only if it can be row-reduced to the identity matrix.
(iv) The rank of a proper submodule of a finite rank free module is strictly less than the rank of the module.

2. Find the Smith normal form of the following matrices

$$\begin{pmatrix} 2 & 4 & 6 \\ 3 & -2 & -3 \\ 2 & 3 & 4 \end{pmatrix} \text{ over } \mathbb{Z}; \qquad \begin{pmatrix} 1+i & 2-i \\ 3+2i & 1+2i \end{pmatrix} \text{ over } \mathbb{Z}[i]; \qquad \begin{pmatrix} 2x^2+3 & 5 \\ x-2 & 2x+3 \end{pmatrix} \text{ over } \mathbb{Q}[x].$$

3. Let $N = \mathbb{C}^3$ be the $\mathbb{C}[x]$-module given by the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(i) What is the rank of N?
(ii) Show that N is cyclic.
(iii) Write N as the direct sum of three non-zero submodules.
(iv) Is N isomorphic to $\mathbb{C}[A]$ as a $\mathbb{C}[x]$-module?
(v) Do your answers to (i), (ii), (iii), (iv) change if we replace $\mathbb{C}$ with $\mathbb{R}$? with $\mathbb{Z}_7$?

4. Find the rank and the invariant factors of the abelian group A with generators a, b, c and relations

$$2a - 16b - 8c = 0, \qquad 4a + 24b + 8c = 0.$$

Find also, in terms of a, b, c, all the elements of A of order 4.

5. Up to isomorphism, how many Abelian groups are there of order (i) 120, (ii) 105, (iii) 360?
(iv) (Optional) For each group of order 120, say how many elements there are of each order.

6. (i) Determine the characteristic and minimal polynomials of the matrix A below.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 4 \\ 1 & 1 & -1 & 3 \end{pmatrix}.$$

(ii) Given your answers to (i), what are the possible rational canonical forms of A? How would you quickly decide which is the correct one?
(iii) Rederive the rational canonical form of A by putting the matrix $xI - A$ into Smith normal form.
(iv) What is the Jordan normal form of A?

7. A non-zero module is said to be *simple* if it has no non-zero proper submodule.
(i) Prove that any simple R-module is isomorphic to a module of the form R/M where M is a maximal ideal.
(ii) Prove *Schur's lemma* which states that any homomorphism $\phi\colon M_1 \to M_2$ of simple R-modules is either zero or a isomorphism.

8. (Optional) (i) Let A be an $n \times n$ matrix over a field F and $\mathbf{v}$, $\mathbf{w}$ be in F^n. Show that there is a monic polynomial $m_{A,\mathbf{v}}(x)$ of least degree such that $m_{A,\mathbf{v}}(A)\mathbf{v} = \mathbf{0}$. Show further that

$$\dim_F Z(\mathbf{v}, A) = \deg m_{A,\mathbf{v}}(x)$$

where $Z(\mathbf{v}, A)$ is the cyclic submodule generated by $\mathbf{v}$.
(ii) Assume that $m_{A,\mathbf{v}}(x)$ and $m_{A,\mathbf{w}}(x)$ are coprime in $F[x]$. Show that

$$m_{A,\mathbf{v}+\mathbf{w}}(x) = m_{A,\mathbf{v}}(x) m_{A,\mathbf{w}}(x)$$

and deduce that $Z(\mathbf{v}, A) + Z(\mathbf{w}, A)$ is a direct sum and also cyclic.
(iii) Deduce that there is a vector $\mathbf{x}$ such that $m_{A,\mathbf{x}}(x) = m_A(x)$.
(iv) Deduce also that $c_A(x) = m_A(x)$ if and only if there is an A-cyclic vector (i.e. a $\mathbf{v}$ in F^n such that $Z(\mathbf{v}, A) = F^n$).