1. Suppose that the evolution of a population can be described by a discrete-time Ricker model of the form
$$N_{t+1} = N_t \exp\left[r\left(1 - \frac{N_t}{K}\right)\right],$$
where $0 < r < 2$.

   (a) Describe the biological interpretation of the model.

   (b) Determine any non-negative steady states and their linear stability.

   (c) Construct a cobweb map the model and discuss the global qualitative behaviour of the solutions.

2. Consider the effect of regularly harvesting the population of a species for which the model equation is
$$N_{t+1} = \frac{bN_t^2}{1+N_t^2} - EN_t := f(N_t; E),$$
where E is a measure of the effort expended in obtaining the harvest, EN_t, and the parameters are such that $b > 2$ and $E > 0$.

   (a) Determine the steady states and hence show that if the effort $E > E_M = (b-2)/2$ no harvest is obtained.

   (b) If $E < E_M$ show by cob-webbing $N_{t+1} = f(N_t; E)$, or otherwise, that the model is realistic only if the population, N_t, always lies between two positive values which should be determined analytically.

   (c) Demonstrate the existence of a tangent bifurcation as $E \to E_M$.

3. Consider a population of annual plants in which seeds are produced at the end of the summer. From these seeds: (i) a proportion survive one winter and, of those that do, a proportion go on to germinate the following spring; (ii) a proportion survive a second winter and, of those that do, a proportion go on to germinate the spring following this second winter; and (iii) no seeds germinate later than this.

   (a) Justify the model
$$N_{t+2} = \alpha\sigma\gamma N_{t+1} + \beta(1-\alpha)\sigma^2\gamma N_t,$$
taking care to interpret each of the parameters biologically.

   (b) By considering the population at the flowering stage (*i.e.* just before new seeds are produced), when there are P_n plants and S_n one-year-old seeds, write the model in terms of a Leslie matrix.

   (c) Show that for the population to thrive the parameters must satisfy
$$\gamma > \frac{1}{\alpha\sigma + \beta(1-\alpha)\sigma^2},$$
and further show that this condition is equivalent to requiring $R_0 > 1$, where R_0 is the basic reproductive ratio and, in this case, can be interpreted as the expected number of offspring produced by an individual during its lifetime that survive to breed, in the absence of any other mortality.

4. Suppose that the predation $P(N)$ on a population $N(t)$ is very fast. A continuous-time model for the evolution of the prey density can be written

$$\frac{\mathrm{d}N}{\mathrm{d}t} = RN\left(1 - \frac{N}{K}\right) - P\left(1 - \exp\left[-\frac{N^2}{\epsilon A^2}\right]\right),$$

where $0 < \epsilon \ll 1$ and R, K, P and A are positive constants.

   (a) Explain the biological interpretation of the different terms in the model.

   (b) If the units of N are density and those of t are time, what are the dimensions of R, K, P, A and ϵ? Hence show that

$$u = \frac{N}{A}, \qquad \tau = \frac{P}{A}t, \qquad r = \frac{AR}{P}, \qquad q = \frac{K}{A},$$

   are non-dimensional.

   (c) Show that the model can be non-dimensionalised to give

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = ru\left(1 - \frac{u}{q}\right) - \left(1 - \exp\left[-\frac{u^2}{\epsilon}\right]\right),$$

   where r and q are positive parameters.

   (d) Demonstrate that there are three possible non-zero steady states if r and q lie in a domain in (r, q) space given approximately by $rq > 4$.

   (e) Could this model exhibit hysteresis? Justify your answer.

5. A continuous-time model for evolution of baleen whale numbers is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\mu N(t) + \mu N(t-T)\left[1 + q\left\{1 - \left(\frac{N(t-T)}{K}\right)^z\right\}\right], \tag{1}$$

where μ, T, q, K and z are positive parameters.

   (a) Explain the biological interpretation of each of the terms in the model.

   (b) Determine the steady states of the model.

   (c) Show that the linear stability of the positive equilibrium is determined by $\mathrm{Re}(\lambda)$ where

$$\lambda = -\mu - \mu(qz - 1)e^{-\lambda T}.$$